VOLUME 37 · NOVEMBER, 1951 NUMBER 11

PROCEEDINGS

OF THE

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Publication Office: Mack Printing Company, Easton, Pa.
Editorial Office: Harvard School of Public Health, Boston 15
Home Office of the Academy: Washington 25, D. C.

Entered as Second-Class matter, June 21, 1920, at the post-office at Easton, Pennsylvania, under the Act of August 24, 1912. Accepted for mailing at a special rate of postage provided for in section 1103, act of October 3, 1917. Authorized June 18, 1920.

Published Monthly Price per Annum, $7.50

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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 37 November 15, 1951 Number 11

ON THE ORIGIN OF THE IRREGULAR SATELLITES

By Gerard P. Kuiper

Yerkes Observatory, University of Chicago

Communicated September 25, 1951

Thirty satellites are known in the solar system. They fall into three classes:

1. The regular satellites.
2. The irregular satellites.
3. The moon.

The regular satellites are the two of Mars, the inner five of Jupiter, the inner seven of Saturn and the five of Uranus, 19 in all. The regular satellites have nearly circular orbits, their motion is direct (in the same sense as the planetary rotation) and the inclination with respect to the planetary equators are all less than 2°. Furthermore, the spacings of these satellites are roughly in a geometrical progression, as is true for the planets around the sun. More accurately, the spacings appear to depend on the masses of the satellites in essentially the same manner as is true for the planetary system; i.e., the systems of regular satellites are homologs of the planetary system.[1] This fact has led[2] to an interpretation of the origin of both the planetary system and of the regular satellites in terms of tidally stable proto-planets and proto-satellites, formed in each case from a disk-shaped nebula by the action of gravitational instability.

The moon is an exceptional object. Its large mass, 1/81 of its primary, indicates that it is not an ordinary satellite. For all other satellites, and for the planets to the sun, the mass ratio is less than 10^{-3}. The lunar composition (density of olivine, 3.3; absence of an iron core) further indicates that the moon was formed as a *twin planet* with the earth.

The ten irregular satellites are listed in table 1. The second column gives the semimajor axis of the satellite orbit a_S in terms of the action radius of the planet, R_A. The latter is given by the interpolation formula[3]

$$\log (R_A/a_P) = 0.318 \log \mu - 0.327, \; 10^{-1} > \mu > 10^{-5}. \qquad (1)$$

Here a_P is the semimajor axis of the planet orbit and $\mu = M_P/(M_\odot + M_P)$,

the fractional mass ratio. For Jupiter, R_A/a_P is 0.0516, for Saturn 0.0352 and for Neptune 0.0204. R_A is the largest distance from the planet to which a satellite may go without being lost to the planet, provided its velocity in the coordinate frame rotating with the planet is zero at this maximum distance. For finite velocities, the maximum distance is less. It is seen that the stability of J VIII is precarious, its apojove being at the distance 0.80 R_A.

The eccentricity e is exceptionally large for Nereid, while Triton (close to the planet) has a circular orbit. The orbital inclinations, i, are given with respect to the planet's equator except where indicated otherwise. The angle of Jupiter's equator being only 3° with respect to its orbit, the six Jupiter satellites have moderately large (20–30°) inclinations with respect to either reference system. Triton's present inclination with respect to Neptune's orbit is 40°, but this value has little physical meaning since it varies (from 40° to 0°) owing to precession of the satellite orbit on the planetary equator. The last column shows the direction of motion: D = direct, R = retrograde. It is seen that there are five satellites in each class.

TABLE 1

NAME	$a8/R_A$	e	$i(°)$	D OR R
J VI	0.29	0.16	29[a]	D
J VII	0.29	0.21	28[a]	D
J X	0.29	0.08	28[a]	D
J VIII	0.58	0.38	32[a]	R
J IX	0.59	0.27	23[a]	R
J XI	0.56	0.21	17[a]	R
Iapetus	0.07	0.03	14;16[a]	D
Phoebe	0.26	0.17	31;5[a]	R
Triton	0.004	0.00	20	R
Nereid	0.06	0.76	5[a]	D

[a] With respect to planet's orbital plane.

There might, perhaps, be some doubt as to whether Iapetus should be included among the irregular satellites. It was classed among them because all clearly regular satellites have $i < 2°$. Quite possibly Hyperion, with $e = 0.104$, is an irregular rather than a regular satellite in spite of its small i value ($<1°$). All clearly regular satellites have $e < 0.02$. The strong perturbations which Hyperion experiences from Titan makes the original nature of its orbit uncertain.

Elsewhere[4] the writer has proposed two alternative explanations for the retrograde satellites: (1) it was found that collisions between the outer parts of consecutive proto-planets can cause retrograde motion of the detached parts with respect to one of the two colliding proto-planets; (2) the decrease of mass on the part of all developing proto-planets will cause the loss of certain satellites formed before the planetary mass reached its ultimate

minimum value. The writer wishes now to withdraw hypothesis (1), as ineffective, and put forward the second hypothesis as an explanation of *all* irregular satellites, retrograde and direct.

The mechanism proposed operates as follows. Let the planet decrease its mass by the factor D after a given satellite is formed. It will be seen below that D may be as large as 10 or even 100. The law of areas will not be affected by this decrease of mass (since it merely depends on the force being central); but the energy of the system, and hence a_S, will be. The areal velocity is $\pi a_S^2/P$ = constant, if P is the period; while Kepler's third law gives $a_S^3/P^2 = cM_S/D$, if M_S is the mass of the planet at the time of satellite formation. Elimination of P gives a_S/D = constant. The ratio a_S/R_A will increase because of two effects: $a_S \sim D$ and R_A, by equation (1), is proportional to $D^{-0.3}$. Therefore

$$a_S/R_A = \text{constant} \cdot D^{1.3}. \qquad (2)$$

The value of D is limited by $D_{\max}$, being the total mass decrease beginning with the early proto-planet stage. $D_{\max}$ is 10^3 for the Earth and Venus, about 20 for Jupiter, 90 for Saturn and 120 for Neptune.[5] D values as large as 10–10^2 may therefore occur for satellites formed rather early in the proto-planet history. Such satellites will be lost unless they formed at very small a_S/R_A values, i.e., very close to the center of the proto-planet. Viscosity of the proto-planet will change the dynamical argument only for satellites embedded in its denser parts; they will move out more slowly than computed by equation (2).

A satellite that has thus been shed by a parent planet will continue to move around the sun in an orbit closely resembling that of the planet. It is expected to be confined approximately to the zone $a_P \pm R_A$. It is improbable that the planet just reached its final (present) mass when the satellite left it; the general case will be one in which the planet continues to lose mass, i.e., one in which its capture cross-section was still large. Sooner or later the lost satellite may collide with the proto-planet and be captured by it. Such capture may result either in direct or retrograde motion around the planet, depending on the geometry of the collision. A collision leading to retrograde motion would offer somewhat more resistance to the body than one leading to direct motion, so that among the recaptured satellites some preference for retrograde orbits is expected. The numbers, 5 and 5, are too small to define the empirical ratio with precision.

Because of the flatness of the proto-planet, no very high inclinations of the recaptured satellites are expected. The two groups of Jupiter satellites might actually represent only *two* captures, with each mass having been split into three parts, not unlike the break-up of meteorites upon entering the earth's atmosphere.

The eccentricities of the recaptured satellites are expected to have been

initially high since as seen from the planet these bodies entered with slightly hyperbolic orbits, transformed into elliptical ones only by friction. Repeated passages through the denser parts of the proto-planets would thereupon have rounded off the orbits. In one case, Triton, the penetration has evidently continued to deeper and deeper layers, causing a very small value of a_S/R_A and a practically circular orbit. Since a large reduction in eccentricity implies a large increase in mass by accretion, the comparatively large mass of Triton is consistent with its low orbital eccentricity. Only the subsequent dissolution of the proto-planet must have prevented Triton (moving against the rotation of the gases) from falling on the planet proper.

After the satellites were recaptured, their new a_S values will have increased again. In principle, the process of loss and recapture could have occurred more than once. When the planetary masses finally stabilized, the recaptured satellites might have been at almost any value of a_S/R_A between 0 and 1. This is actually the case (table 1).

It is sometimes assumed that the so-called Trojan asteroids were formed in the asteroid ring ($2 < a < 4$ astr. units) and moved to their present a values ($\simeq 5.2$) by perturbations of Jupiter. This assumption is contrary to the conclusion based on Jacobi's integral for the system Sun-Jupiter-Asteroid, namely that all asteroids with $a < 4.2$ and with small orbital eccentricities (e) and inclinations (i) must remain there.[6] For finite e and i values the stability limit is somewhat less than 4.2 astr. units.

Only fourteen asteroids are known with $a > 4.0$: (1) Thule, $a = 4.25$, $e = 0.06$ and $i = 2°$ (with respect to the ecliptic) is evidently a border-line case, though at present somewhat outside the stability limit; (2) twelve Trojans, $a = 5.1$–5.2; and (3) Hidalgo, $a = 5.79$, $e = 0.65$, $i = 43°$. *It is suggested that the Trojans and Hidalgo are shedded satellites of Jupiter.* Hidalgo can come close to Jupiter; this fact, and the large e and i values, support the view that Hidalgo's orbit resulted from a close approach to Jupiter. It is of interest, in this connection, that the absolute magnitudes of the six irregular satellites of Jupiter, of the twelve Trojans and of Hidalgo are comparable; they correspond to diameters of the order of 50 km.

If the Trojans are in fact shedded satellites, they are expected to be nearly spherical and show little variation in light owing to rotation (≤ 0.1–0.2 mag.). If, instead, they are typical asteroids, they are expected to show light variations of about 0.4 mag. amplitude and with periods between 5 and 10 hrs.; such results were found empirically for a sample of the asteroids.[7] It is hoped that photometric tests now scheduled will decide between these alternatives.

[1] Kuiper, G. P., *Astrophys. J.*, **109**, 308 (1949).

[2] Kuiper, G. P., these PROCEEDINGS, **37**, 1 and 383 (1951).

[3] This equation is equivalent to, but somewhat more accurate than, equation (1') on p. 385 of this Volume. A full discussion of its derivation will be given elsewhere.

The formula is precise for $\mu = 10^{-2}$ and 10^{-4} and leads to errors of 1% in R_4 at $\mu = 10^{-1}$, 10^{-3} and 10^{-5}.

[4] *Astrophysics*, edited by J. A. Hynek, McGraw-Hill Book Co., Inc., New York, N. Y., 1951, p. 405.

[5] These PROCEEDINGS, **37**, 386, table 1, and 389 (1951).

[6] E.g., C. L. Charlier, *Die Mechanik des Himmels*, Vol. 2, W. de Gruyter, Berlin, 1927, Sec. Ed., p. 293.

[7] Harris, D. L., and Kuiper, G. P., *Astrophys. J.* (in press).

THE ORIGIN OF PHOSPHORUS IN T5 BACTERIOPHAGE OF ESCHERICHIA COLI

BY LOUIS W. LABAW

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Communicated by R. W. G. Wyckoff, September 28, 1951

The use of isotopes has given considerable information concerning the origin of chemical components of *Escherichia coli* bacteriophages.[1-5] Our recent studies employing radioactive phosphorus have shown that, though various percentages of the phosphorus they contain originate in the host bacterium before infection, all seven bacteriophages of the T system obtain from the host substantially the same absolute amount of phosphorus per infectious unit.[6] In an effort to interpret this equality a more detailed study of the origin of the phosphorus in T5 bacteriophage has now been carried out. These results show that the bacterial phosphorus incorporated into T5 bacteriophage has been taken up by the host, *E. coli*, within the two minutes before infection. The mean generation time of *E. coli* in the medium used is 23 minutes and the burst time for T5 bacteriophage is 40 minutes. These time relations indicate that, contrary to conclusions drawn from work with other T bacteriophages, the bacterial phosphorus going into the T5 bacteriophage does not come from the desoxyribonucleic acid of the host bacteria.

Methods.—The *E. coli* cultures were taken from the same B strain used in previous studies with radioactive phosphorus; they were maintained as described in previous papers.[7, 8] The T5 bacteriophage was that used in a previous investigation[6] and was originally obtained from Professor M. H. Adams. It was found necessary to supplement the tryptose broth with $CaCl_2$ to give the $10^{-3}M$ concentration needed for a satisfactory yield of T5.[9] The concentration of radioactivity in broth labeled with P^{32} was adjusted to give approximately 10^6 counts per minute per ml. and the data were all normalized to this figure.

The T5 bacteriophage was purified by sedimenting once from the filtered lysates and twice more from 0.85% NaCl. The three ultracentrifugations, at an average of 12,000 g. for 30 minutes in the 1.58-cm. diameter tubes of a No. 40 Spinco angle head, were interspersed with low-speed clearings in an angle centrifuge.

The number of infectious T5 units was determined by counting plaques on horse meat agar plates inoculated with mixtures of *E. coli* and appropriate dilutions of T5 after incubating the plates for 18 hrs. at 37°C. For these titrations the diluted T5 suspensions were allowed to stand with *E. coli* for 15 minutes to facilitate adsorption before the mixture was spread on horse meat agar and the plates dried in a 43°C. oven. Surviving bacteria which presumably had failed to adsorb particles of the bacteriophage

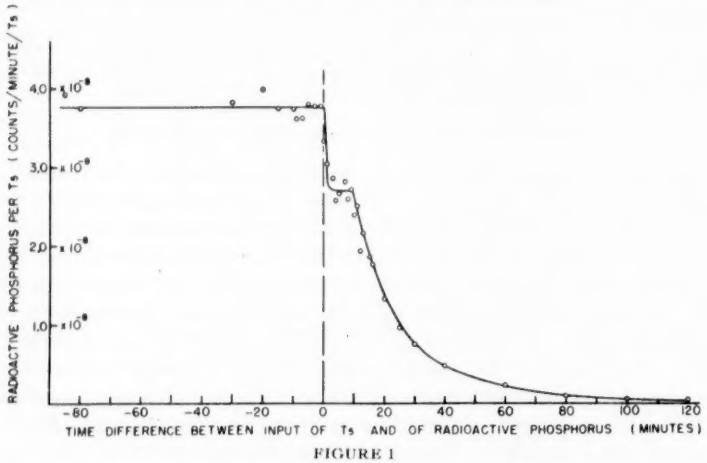

FIGURE 1

The amount of labeled phosphorus incorporated into the T5 bacteriophage of *E. coli* as a function of the time of addition of P^{32} to the culture. These experimental points are averages where more than one value was measured (see figure 2).

were determined from dilution platings of the *E. coli*-T5 mixture on tryptose phosphate agar. This tryptose phosphate agar does not contain enough calcium to support T5 multiplication and hence an accurate measure of the uninfected bacteria can be obtained if the plates are dried rapidly.

As usual the radioactive assay was made by measuring dried pellets of bacteriophage under an end-window counter. By making these pellets from an aliquot fraction of the final T5 suspension, their radioactivity is correlated with the number of infectious particles they contain.

Results.—In order to get information about the state of the phosphorus that T5 bacteriophage inherits from its host, the inorganic phosphorus in

the culture medium was labeled at different times before and after addition of infecting phage to the culture of growing bacteria. Small volumes (130 ml.) of the parent *E. coli*, or *E. coli* + T5, cultures were labeled at various times. All the bacterial cultures had bacteriophage added in the ratio of 10 to 15 per bacterium when the bacterial count had reached 7×10^7 per ml. Lysis was complete in two hours with an average burst size of about 200. The T5 bacteriophage in each 130-ml. volume was then purified. The number of counts per minute per infectious unit was calculated from the radioactive assay and two independent plaque count assays involving the final T5 pellet. This was plotted as a function of the time difference between the introduction of P^{32} and the addition of T5. As shown in figure 1 averages

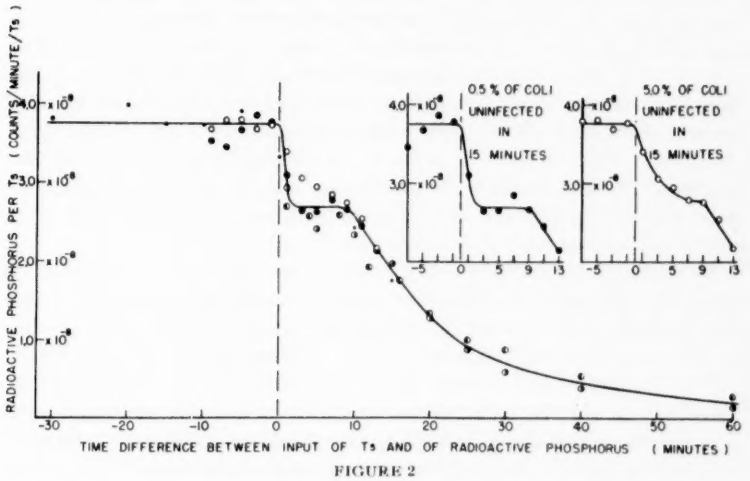

FIGURE 2

An expansion of part of the curve in figure 1, showing all the experimental points in the region −30 to 60 minutes. The two small graphs show how the absorption rate affects the results. The data from each separate experiment are plotted with a different symbol.

have been plotted where more than one experimental point was obtained for a given time difference; the entire plot is a composite of points from five experiments of twelve points each. Figure 2 is a similar curve containing all points derived from experiments in which P^{32} was introduced between 30 minutes before the inoculation with bacteriophage to 60 minutes thereafter.

These figures show that the amount of radioactive phosphorus incorporated into the T5 bacteriophage is independent of the time of labeling the bacteria up to about two minutes before infection with T5.

The radiophosphorus in the bacteria, on the other hand, is dependent upon the time of labeling since the bacterial phosphorus doubles every 23 minutes during the part of the logarithmic growth phase used in these experiments.[7] Thus at the time of infection, about 90% of the *E. coli* phosphorus had been derived from the medium during the previous 85 minutes, whereas only 13% had come from the medium in the 5-minute interval before infection. The lack of dependence of the bacterial phosphorus in T5 bacteriophage on the percentage of bacterial phosphorus labeled at the time of infection indicates that this phosphorus can scarcely have been derived from the desoxyribonucleic acid of the bacteria.

The right-hand parts of the curves in figures 1 and 2, where the labeling has been introduced after infection, show first a relatively sharp drop to the level given in the earlier paper[6] as the uptake after infection by T5 of inorganic phosphorus from the medium. Following this for the first nine minutes after infection, there is practically no additional phosphorus uptake from the medium; but this level state is followed by a further drop continuing with reduced slope throughout the two hours of the experiment. The low uptake of radioactivity after the end of the burst time attests to the lack of appreciable adsorption of P^{32} on the bacteriophage, especially since the bacteriophage purified for these last experimental points remained in the radioactive labeled lysates for over 24 hrs.

The sharp drop and subsequent leveling off of the curve in the interval from 0 to 10 minutes after addition of bacteriophage seems to indicate that adsorption of T5 blocks any further utilization for bacteriophage production of phosphorus the bacteria can take up during this period. The *E. coli*-T5 complex begins to take up phosphorus from the medium for bacteriophage production after about 10 minutes, and this continues until lysis occurs. The role of T5 adsorption in arresting uptake by the bacteria is shown by the difference between the curves in the two small diagrams of figure 2. Here the experiment involving slower adsorption gave higher phosphorus values for P^{32} additions in the interval from zero to ten minutes;[10] this would be expected if the bacteria take up phosphorus normally until infected.

The ordinates of figures 1 and 2 may be approximately converted to absolute phosphorus contents by using the ratio of (chemical) inorganic phosphorus to radioactive phosphorus per ml. of culture medium. In these experiments this was measured as:

$$\frac{104\gamma\text{P/ml.}}{10^6\text{ counts/min./ml.}} = 1.04 \times 10^{-3} \quad \frac{\gamma\text{P}}{\text{count/min.}}$$

Hence the phosphorus content per T5 bacteriophage would be about 3.9×10^{-17} g. in agreement with the previous measurement.[6]

Discussion.—It is tempting to use the new experimental data to speculate

on the nature of the bacterial phosphorus contributed to bacteriophage. The present experiment shows that the bacterial phosphorus reaching T5 is taken from the medium no more than two minutes before infection. It therefore can scarcely be the phosphorus in the bacterial nucleic acids. This is substantiated by the following calculation. The bacterial phosphorus contributed to T5 is equivalent to all of the normal phosphorus uptake by uninfected bacteria for a period of about $1^1/_2$ minutes. This figure is obtained by dividing the product of bacterial phosphorus per T5 bacteriophage, 1.2×10^{-17} g. P/T5, and the average burst size of 200, by the uptake of phosphorus per minute by similarly cultured *E. coli* [1.6×10^{-15} g. P/bacterium/min.].[8] If then one computes the normal bacterial DNA production per minute taking the DNA phosphorus as 13% of the total bacterial phosphorus[11] one finds that this requires an uptake of only 2×10^{-16} g. P per minute. This, for two minutes, would only be 17% of the bacteriophage phosphorus that is contributed by the bacteria.

Acknowledgment.—This work was carried out in the laboratory of Dr. Ralph W. G. Wyckoff whom the author wishes to thank for many helpful discussions.

The radioactive phosphorus used in this investigation was supplied by the Oak Ridge National Laboratory on allocation from the Isotopes Division, United States Atomic Energy Commission.

Summary.—The amount of labeled phosphorus incorporated into the T5 bacteriophage of *Escherichia coli* has been measured as a function of the time of addition of radioactive phosphorus to the medium both before and after infection. The results show that the part of the T5 bacteriophage phosphorus which comes from the bacteria before the addition of the virus to the broth culture is taken from the medium in the two minutes preceding infection. This precludes the bacterial desoxyribonucleic acid as the major source of the bacterial phosphorus contributed to T5 bacteriophage.

[1] Cohen, S. S., *J. Biol. Chem.*, **174,** 295 (1948).

[2] Barry, J. M., Banks, M. G., and Koch, A. L., *Fed. Proc.*, **9,** 148 (1950).

[3] Kozloff, L. M., and Putnam, F. W., *J. Biol. Chem.*, **182,** 229 (1950).

[4] Evans, E. A., Jr., Putnam, F. W., and Miller, D., *Fed. Proc.*, **10,** 181 (1951).

[5] Kozloff, L. M., Knowlton, K., Putnam, F. W., and Evans, E. A., Jr., *J. Biol. Chem.*, **188,** 101 (1951).

[6] Labaw, L. W., *J. Bact.*, **62,** 169 (1951).

[7] Labaw, L. W., Mosley, V. M., and Wyckoff, R. W. G., *Ibid.*, **59,** 251 (1950).

[8] Labaw, L. W., Mosley, V. M., and Wyckoff, R. W. G., *Ibid.*, **60,** 511 (1950).

[9] Adams, M. H., *J. Immunol.*, **62,** 505 (1949).

[10] A single smooth curve is incapable of describing both sets of data in this interval since a paired comparison of all the observations taken at the same times for the two adsorption rates yields a mean difference of 0.3×10^{-8} counts/min./T5 with a standard error of 0.09 giving a significant difference.

[11] Kozloff, L. M., and Putnam, F. W., *J. Biol. Chem.*, **181,** 207 (1949).

THE EXTENDED ONSAGER EQUATION

By Edward G. Baker and Charles A. Kraus

Metcalf Research Laboratory, Brown University

Communicated September 18, 1951

The extended equation of Onsager[1] may be written:

$$\Lambda_0' - \Lambda_0 = BC + DC \log C, \tag{1}$$

where

$$\Lambda_0' = (\Lambda + \beta\sqrt{C})/(1 - \alpha\sqrt{C}), \tag{2}$$

B and D are empirical constants and α and β are the usual constants of the simple Onsager equation. If both B and D are zero, values of Λ_0' as calculated from the experimental data will be equal to Λ_0 within the limit of experimental error. The equation thus expresses the deviation from the simple Onsager equation as a function of concentration. If D is zero and B is positive, we have the equation of Shedlovsky[2] which holds at lower concentrations for a considerable number of salts. If ion association occurs, the deviation curve is still linear but B becomes negative if association is marked.

For a considerable number of salts, particularly of potassium, investigators have obtained a better fit to the experimental data by employing equation (1) as written.[3] However, up to now, measurements have not been carried out with sufficient precision and analyzed in such manner as to delineate the form of the deviation curve nor the precision with which the equation may be fitted to the experimental data.

The plot of equation (1) passes through a minimum and is tangent to the axis of ordinates at $C = 0$. To find the condition for the minimum we set $d(\Lambda_0' - \Lambda_0)/dC = 0$; solving the resulting equation and converting to common logarithms, we obtain:

$$0.43 + \log C_m = -B/D \tag{3}$$

where C_m is the concentration at the minimum. Above the minimum, there is a concentration, C^*, where BC and $DC \log C$ are equal and of opposite sign. At this concentration, $\Lambda_0' = \Lambda_0$. We thus have

$$\log C^* = -B/D. \tag{4}$$

Combining (3) and (4) we have

$$0.43 + \log C_m = \log C^*$$

or

$$C^* = 2.7\, C_m \tag{5}$$

If the concentration of the minimum can be determined, it is thus possible to evaluate Λ_0 without graphical extrapolation since $\Lambda_0' = \Lambda_0$ at concentration C^*.

The value of D is most readily obtained by graphical means. Let us transfer the constant Λ_0 to the right-hand side of equation (1) and let us write $B/D = n$. Then equation (1) becomes

$$\Lambda_0' = \Lambda_0 + D(n + \log C)C. \tag{6}$$

Making use of equation (4), equation (6) may be written in the simple and convenient form:

$$\Lambda_0' = \Lambda_0 + DC \log C/C^* \tag{7}$$

If we plot values of Λ_0' against values of $C \log C/C^*$, we should obtain a straight line whose slope is equal to D and whose intercept with the ordinate corresponding to $\log C/C^* = 0$ yields the value of Λ_0.

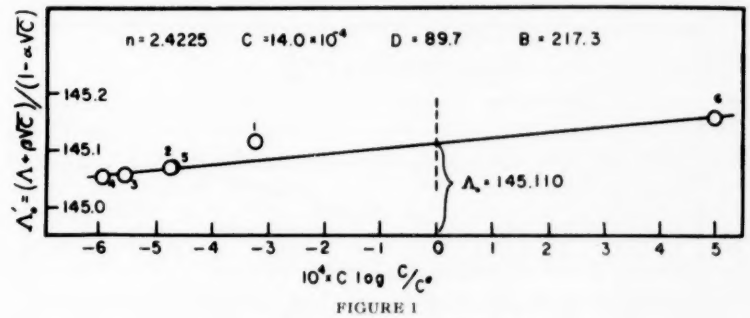

FIGURE 1

Plot for the evaluation of D and λ_0.

We present an analysis of our results with potassium nitrate at concentrations below $5 \times 10^{-3}N$. From a plot of Λ_0' vs. C, C_m was found to be $14 \times 10^{-4}N$. This yields $C^* = 37.8 \times 10^{-4}N$. At this concentration, Λ_0' (from a Λ_0' vs. C plot) is 145.12 and this is, therefore, the value of Λ_0. For $B/D = n = -\log C^*$, we obtain the value of 2.4225.

With this value of n, a plot was constructed of Λ_0' against $C \log C/C^*$ (equation (7)). This plot is shown in figure 1. The points are numbered in order from the most dilute to the most concentrated solution. All points but the first one lie on a straight line with a deviation less than 0.005%. This plot is very sensitive; the departure of the first point from the straight line corresponds to an experimental error of only 0.028%. For Λ_0 we obtain 145.110 as against 145.120 as found above. The lower value is the more reliable since it was obtained by means of a linear interpolation.

From the plot we obtain the values: $D = 89.7$, $\Lambda_0 = 145.110$, $B = nD =$

217.3. With the aid of these constants, we have the numerical conductance equation:

$$\Lambda_0' = 145.110 + 217.3C + 89.7C \log C \qquad (8)$$

Using this equation, we have computed values of Λ_0' at the measured concentrations. The results are shown in the following table.

POTASSIUM NITRATE AT 25°

POINT NUMBER	$C \times 10^4$	Λ_0' (exp.)	Λ_0' (calc.)	$\Delta\Lambda_0'$
1	2.89432	145.116	145.081	+0.035
2	5.84448	145.071	145.067	+0.004
3	8.76103	145.056	145.060	−0.004
4	11.6750	145.052	145.057	−0.005
5	24.1318	145.071	145.067	+0.004
6	48.0272	145.158	145.155	+0.003

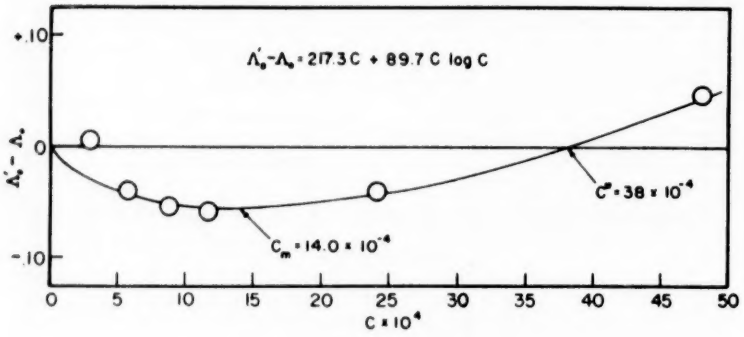

FIGURE 2

Plot of equation (1).

In figure 2, the smooth curve represents the numerical equation; the circles represent experimental values of Λ_0'; their radii correspond to an error of 0.01 Λ-unit in the experimental conductance values. With the exception of the first point, the deviation of the experimental values from the equation is less than 0.005%. It would appear, therefore, that for potassium nitrate, the extended Onsager equation reproduces the observed conductances very precisely.

The deviation curves of all potassium salts measured (chloride, bromide, iodide, nitrate and bromate) exhibit minima, as also do cesium picrate and chloride and sodium picrate. The plot for sodium iodide is linear with positive slope. Quaternary ammonium salts of large anions yield linear plots

with positive slope but their salts with small anions yield linear plots with negative slope.

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CONFIGURATIONS OF POLYPEPTIDE CHAINS WITH FAVORED ORIENTATIONS AROUND SINGLE BONDS: TWO NEW PLEATED SHEETS

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Communicated September 4, 1951

In recent papers we have described several configurations of polypeptide chains with interatomic distances, bond angles, and other structural features as indicated by the studies in these Laboratories of the structure of crystals of amino acids, simple peptides, and related substances, and have presented evidence for their presence in synthetic polypeptides, fibrous proteins, and globular proteins.[1-9] The requirements that we have imposed for a satisfactory polypeptide configuration, in addition to the correct bond distances and bond angles, are that each amide group be planar, with either the cis configuration or the trans configuration about the C′—N bond (which has nearly 50 per cent double-bond character), and that each carbonyl and imino group (except for proline or hydroxyproline residues) be involved in the formation of a hydrogen bond with N—H···O distance approximately 2.8 A and with the oxygen atom nearly on the N—H axis. The only other structural parameters involved in a configuration of polypeptide chains are the orientations around the N—C and C—C′ single bonds. In the following paragraphs we discuss the question of the relative stability of structures with different values of these orientational parameters.

We are interested in the potential function for orientation around a single bond between the α carbon atom, which forms three single bonds in addition to the bond under consideration, and either the nitrogen atom, which forms a single bond to its hydrogen atom and a bond with somewhat less than 50 per cent double-bond character to the amide carbon atom, or the carbon atom C′, which forms a bond with oxygen which has somewhat more than 50 per cent double-bond character and a bond with nitrogen with

somewhat less than 50 per cent double-bond character. We shall discuss only polypeptides containing residues with the L configuration. The potential function would be expected to have six minima, approximately 60° apart, and six maxima. If the three single bonds on the α carbon atom were equivalent and the two bonds formed by the other atoms were equivalent, as for example in nitromethane, $H_3C—NO_2$, the potential function would have the form shown in figure 1 (*a*), with equivalent minima and equivalent maxima. For propylene, $H_3C—CH=CH_2$, the potential function has either the form shown in figure 1 (*b*) or that in figure 1 (*c*); either the minima are divided into non-equivalent sets of three each, or the maxima are so divided. Curve *1b* corresponds to the situation in which the stable configurations are those with one of the bonds of the α carbon atom coplanar

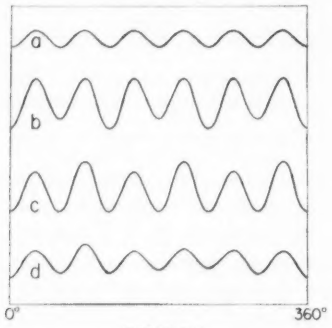

FIGURE 1

Forms of the potential function for orientation around a single bond connecting a tetrahedral carbon atom to a planar group.

with the $CH=CH_2$ group (the two sets representing the configuration cis to the double bond and trans to the double bond, respectively) and curve *1c* represents the situation in which the stable configurations are those with one of the single bonds of the α carbon atom in a plane normal to the plane of the $CH=CH_2$ group.

The height of the potential hump for nitromethane has been independently determined by three investigators, by measurements of heat capacity and of entropy,[10-12] their average value being 0.6 kcal. mole^{-1} (Pitzer and Gwinn, 0.8 kcal. mole^{-1}; DeVries and Collins, 0.6 kcal. mole^{-1}; Jones and Giauque, 0.5 kcal. mole^{-1}). For propylene the height of the potential hump is larger; the value 0.8 kcal. mole^{-1} or less originally suggested by Pitzer[13] has been replaced by the value 2.1 kcal. mole^{-1}, which is supported by several other investigators.[14-17] In neither case have the thermodynamic

data provided a choice between the alternatives represented by curves *1b* and *1c*. For the bonds that we are considering all of the maxima and all of the minima would be different, and the potential function would be expected to have a form such as that shown by curve *1d*.

It seems reasonable to expect that the average energy differences of the maxima and minima for an amide would be somewhere between 0.6 kcal. mole^{-1} and 2.1 kcal. mole^{-1}, perhaps about 1.0 kcal. mole^{-1}, with considerably smaller differences among the minima themselves. This energy quantity, 1 kcal. mole^{-1}, is sufficiently large to cause configurations represented by the minima in the curve to be significantly favored over those represented by the maxima. On the other hand, in crystals of amino acids and related substances in which hydrogen bonds, with energy as great as 8 kcal. mole^{-1}, are formed the metrical requirements might be such that the most stable structure, involving the best conditions for hydrogen-bond forma-

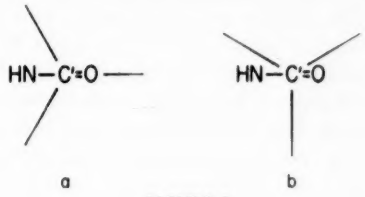

FIGURE 2

A diagrammatic representation of two possible stable orientations of the bonds of the α-carbon atom with respect to the plane of the amide group. The αC atom is directly beneath C'; an adjacent single bond (tapered lines) of the αC atom lies either (*a*) in the plane of the amide group or (*b*) in the plane perpendicular to it.

tion, is one in which the molecule is distorted by rotation around the single bonds to the α carbon atom into a configuration of instability with respect to the orientational potential functions. With proteins in an aqueous environment the effective energy of hydrogen bonds is not so great, inasmuch as the difference between the energy of the system with N—H$\cdots$O hydrogen bonds surrounded by water and a system with the N—H group and the O atom forming hydrogen bonds with water molecules may be no more than about 2 kcal. mole.$^{-1}$ Hence the favored orientations around the bonds to the α carbon atoms may be of more frequent occurrence in proteins than in simple amides.

The first question to be answered is whether the adjacent single bond on the α carbon atom lies in the plane of the amide group or in the plane perpendicular to this plane (Fig. 2). The evidence from the known structures

of simple substances is not conclusive, but indicates that the stable configuration is the coplanar one. In N-acetylglycine[18] the bond from the α carbon atom to the carboxyl carbon is very nearly coplanar with the amide group, the carboxyl carbon lying cis to the N—H group. In β-glycylglycine[19] the same configuration is shown by the carboxyl carbon atom and the amide group. In addition, in each of these two substances the carboxyl group itself lies in the same plane as the amide group (to within 0.03 A in β-glycylglycine), so that the single bond from the amide nitrogen atom to the α carbon atom has assumed a coplanar orientation with respect to the carboxyl group. In the benzyl penicillinate ion[20] the bond between the carbon atom of the benzene ring and the α carbon atom adjacent to the carbonyl carbon atom of the amide group is coplanar with the amide group, and trans to the C—O group. There is also a carbon-carbon single bond involving the α carbon atom attached to the nitrogen atom of the amide group that is coplanar with the amide group, and cis to the N—H group; the significance of this orientation is decreased, however, by the fact that this carbon-carbon bond is part of a four-membered ring. In β-glycylglycine, on the other hand, the amino nitrogen atom is rotated by 27° out of the plane of the amide group. We consider it likely that this corresponds to a strained configuration, and that the larger amount of evidence indicating the stability of the orientation corresponding to a coplanar arrangement of one of the single bonds of the α carbon atom and the amide group is to be accepted.

Further evidence is provided by the structures of synthetic polypeptides and proteins. The most common configuration of polypeptide chains seems to be that of the 3.7-residue helix, which we have proposed as representing the structure of synthetic polypeptides,[4] hair, horn, quill, muscle, and related fibrous proteins,[7] and hemoglobin;[9] valuable confirmation of the assignment of this structure to the substances has been provided by Perutz's discovery that the 1.5 A reflection corresponding to the fiber-axis length per residue, which had been observed for porcupine quill, can also be obtained from the plane normal to the helical axis for the other substances.[21] In this helix the C—H bond and the C—R bond (that is, the bond to the β carbon atom of the R group) lie within a few degrees of the plane of an adjacent amide group. In the three-chain structure that we have proposed for collagen and gelatin[8] five of the six C—H and C—R bonds lie very near the planar configuration, the other one being rotated through nearly 30°, presumably into the unfavored configuration to which it is constrained by the nature of the structure.

Accepting the conclusion indicated by the preponderance of the evidence that the minima for the rotational potential functions for the two bonds connecting the α carbon atom with adjacent atoms in the polypeptide chain correspond to coplanar configurations, we give an exhaustive discussion in

the following paragraphs of configurations of this type in which all of the amide groups are of the trans type and are equivalent.

There are six favored orientations about the C—N bond, which we may describe as CH trans to NH, CR trans to NH, CC′ trans to NH, CH cis to NH, CR cis to NH, and CC′ cis to NH. Similarly, we may describe the six favored orientations around the C—C′ bond as CH trans to CO, CR trans to CO, CN trans to CO, CH cis to CO, CR cis to CO, and CN cis to CO. There are accordingly 36 different configurations of the polypeptide chains of L-amino acid residues corresponding to these favored orientations; all 36 configurations are different. They are tabulated below in four groups:

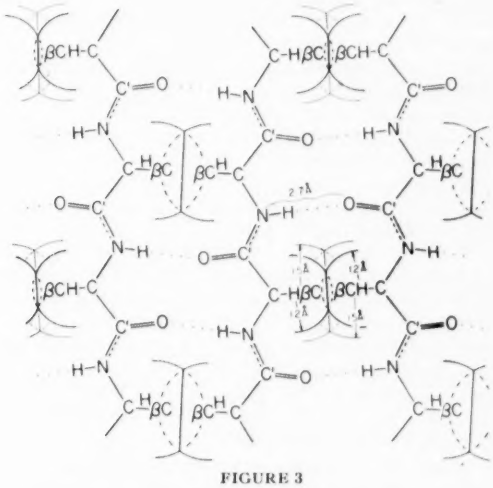

FIGURE 3

A diagrammatic representation of a hydrogen-bonded layer structure of polypeptide chains with alternate chains oppositely oriented, showing steric hindrance between β-carbon atoms of adjacent chains.

trans-trans, cis-cis, trans-cis, and cis-trans. It might be anticipated that the configurations in which CH, CR, or CC′ (or CN) is trans to the bond with the greater amount of double-bond character, of the two bonds once removed, would be either more stable or less stable than the configurations in which one of these bonds is cis to the bond with the greater amount of double-bond character. Accordingly we would expect that either the nine cis-trans structures or the nine trans-cis structures would be somewhat more stable than the other 27 structures; however, the differences in stability are presumably small enough to be unimportant in comparison with the

TABLE 1

CONFIGURATIONS OF EQUIVALENT TRANS AMIDE GROUPS WITH FAVORED ORIENTATIONS ABOUT C—N AND C—C′ BONDS

A. Trans-trans Configurations:
   1. CH trans to NH, CH trans to CO. Pitch 6.68 A, 2 residues per turn. Lateral CO and NH suited to forming the two pleated sheets described below.
   2. CR trans to NH, CR trans to CO. Like 1. Suited to forming the two pleated sheets, but prevented (except for polyglycine) by steric hindrance of R groups.
   3. CC′ trans to NH, CN trans to CO. Ruled out by steric hindrance of NH and O.
   4. CH trans to NH, CR trans to CO. Pitch 6.2 A, 4 residues per turn. Lateral-diagonal CO and NH, suited to forming a tetragonal framework with $a = 4.8$ A, but prevented by steric hindrance of R groups.
   5. CR trans to NH, CH trans to CO. Like 4.
   6. CH trans to NH, CN trans to CO. Ruled out by steric hindrance of NH and N of adjacent group.
   7. CR trans to NH, CN trans to CO. Like 6.
   8. CC′ trans to NH, CH trans to CO. Ruled out by steric hindrance of O and O.
   9. CC′ trans to NH, CR trans to CO. Like 8.

B. Cis-cis Configurations:
   10. CH cis to NH, CH cis to CO. Ruled out by steric hindrance of NH and O.
   11. CR cis to NH, CR cis to CO. Like 10.
   12. CC′ cis to NH, CN cis to CO. Pitch 7.30 A, 2 residues per turn. Lateral CO and NH, suited to forming the planar sheet with alternate chains reversed in direction, but prevented (except for polyglycine) by steric hindrance of R groups.
   13. CH cis to NH, CR cis to CO. Pitch 5.5 A, 3.7 residues per turn. Axial CO and NH, suited to forming the α helix with intramolecular hydrogen bonds.
   14. CR cis to NH, CH cis to CO. Like 13; the α helix.
   15. CH cis to NH, CN cis to CO. Ruled out by steric hindrance of O and O.
   16. CR cis to NH, CN cis to CO. Like 15.
   17. CC′ cis to NH, CH cis to CO. Pitch 10.4 A, 4 residues per turn. Lateral CO and NH, with H of NH near axis, and CO projecting out. Not suited to formation of intramolecular or intermolecular hydrogen bonds (except possibly for polyglycine only, in a tetragonal framework with $a = 4.8$ A).
   18. CC′ cis to NH, CR cis to CO. Like 17.

C. Trans-cis Configurations:
   19. CH trans to NH, CH cis to CO. Pitch 5.2 A, 4.8 residues per turn. Diagonal CO and NH, with NH directed toward axis, CO outward. Not suited to formation of hydrogen-bonded sheet or framework.
   20. CR trans to NH, CR cis to CO. Like 19.
   21. CC′ trans to NH, CN cis to CO. Ruled out by steric hindrance of O and O.
   22. CH trans to NH, CR cis to CO. Pitch 8.0 A, 2.8 residues per turn. Not suited to formation of hydrogen-bonded sheet or framework.
   23. CR trans to NH, CH cis to CO. Like 22.
   24. CH trans to NH, CN cis to CO. Pitch 9.6 A, 2.7 residues per turn. Lateral CO and NH, not suited to formation of hydrogen-bonded sheet or framework.
   25. CR trans to NH, CN cis to CO. Like 24.

26. CC′ trans to NH, CH cis to CO. Ruled out by steric hindrance of O and C′ of adjacent amide group.
27. CC′ trans to NH, CR cis to CO. Like 26.

D. Cis-trans Configurations:

28. CH cis to NH, CH trans to CO. Pitch 5.2 A, 4.8 residues per turn (similar to 19). Not suited to formation of hydrogen-bonded sheet or framework.
29. CR cis to NH, CR trans to CO. Like 28.
30. CC′ cis to NH, CN trans to CO. Ruled out by steric hindrance of NH and NH.
31. CH cis to NH, CR trans to CO. Pitch 7.6 A, 2.7 residues per turn. Lateral CO and NH, not suited to formation of hydrogen-bonded sheet or framework.
32. CR cis to NH, CH trans to CO. Like 31.
33. CH cis to NH, CN trans to CO. Ruled out by steric hindrance of NH and adjacent amide group.
34. CR cis to NH, CN trans to CO. Like 33.
35. CC′ cis to NH, CH trans to CO. Pitch 9.5 A, 2.7 residues per turn. Lateral CO and NH. Not suited to formation of hydrogen-bonded sheet or framework.
36. CC′ cis to NH, CR trans to CO. Like 35.

differences due to other structural features. (Note that the orientation trans to CO is trans to the bond with the larger amount of double-bond character, whereas the orientation cis to NH is trans to the bond with the larger amount of double-bond character.)

In table 1 we give for each of the 36 structures, all of which are helical, the pitch of the helix (axial length per turn), the number of residues per turn, the approximate orientation of the CO and NH groups (relative to the helical axis—axial, diagonal, or lateral), and the suitability of the structure to forming hydrogen bonds. Hydrogen bonds may be intramolecular or intermolecular. In the latter case either a hydrogen-bonded sheet or a hydrogen-bonded three-dimensional framework may be formed. A sheet can be formed only when the helix has a twofold screw axis (or a onefold screw axis, which does not occur), and a three-dimensional framework, under our requirements of equivalence of all of the amide groups, only when the helix has a threefold, fourfold, or sixfold screw axis. In describing the configurations we have allowed deviations of a few degrees from the favored orientations around the single bonds, in order to permit the formation of the suitable hydrogen-bonded structures.

From table 1 we see that the only configurations of interest are 1, 12, 13, and 14. The last two configurations represent the α helix, with two different sets of positions of the side chains. Configuration 12, a completely extended polypeptide chain, with identity distance 7.30 A, is suited to the formation of a planar sheet, as shown in figure 3, except that steric hindrance between adjacent chains prevents the formation of this sheet in case that the side chain R is anything but a hydrogen—that is, this planar sheet could be formed only by polyglycine.

The remaining configuration, 1, is of great interest—it seems likely that it is present in many proteins with the β-keratin configuration. The predicted identity distance along the chain axis is 6.68 A, corresponding to 3.34

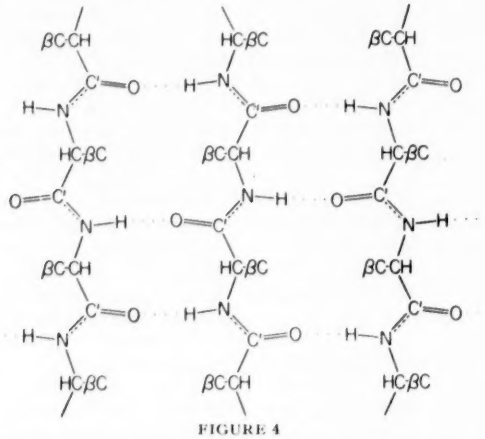

FIGURE 4

A diagrammatic representation of the antiparallel-chain pleated sheet structure.

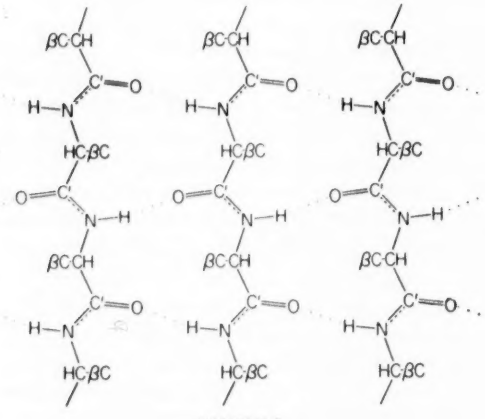

FIGURE 5

A diagrammatic representation of the parallel-chain pleated sheet structure.

A per residue. This agrees very well with the residue length reported by Astbury and Street for proteins with the β-keratin structure, 3.32 A.[22]

We have previously suggested for β keratin a pleated-sheet configuration

of polypeptide chains in which each residue is converted into the following one by the operation of a glide plane of symmetry. With this configuration the orientation around the bonds N—C and C—C′ is not that which we consider to be stable—the stable orientations lead to a residue length in the chain of only 2.84 A, which is not found experimentally for any protein of the β-keratin type. We now think that it is unlikely that this pleated sheet is an important configuration for proteins.

We have found that polypeptide chains with the configuration 1 of Table 1 and residue length 3.34 A can be arranged to form sheets with lateral hydrogen bonds in either one of two ways. In the first way alternate chains in the sheet have opposite directions. The sheet is similar to the first pleated sheet that we described in having α carbon atoms at the same level

TABLE 2

ATOMIC COORDINATES FOR THE ANTIPARALLEL-CHAIN PLEATED SHEET AND PARALLEL-CHAIN PLEATED SHEET

Antiparallel-Chain Pleated Sheet:

Atomic coordinates: $x, y, z;\ \bar{x}, {}^1/_2 + y, \bar{z};\ {}^1/_2 - x, \bar{y}, z;\ {}^1/_2 + x, {}^1/_2 - y, \bar{z}$.

a_0 = 9.46 A

b_0 = 6.68 A (direction of chain axis)

c_0 = 1 A (arbitrary)

Parallel-Chain Pleated Sheet:

Atomic Coordinates: $x', y, z;\ \bar{x}', {}^1/_2 + y, \bar{z}$.

a_0 = 4.73 A

b_0 = 6.68 A

c_0 = 1 A

	x	y	z	x'
C	0.009	0.000	−0.94	0.018
N	−0.036	0.184	−0.25	−0.072
C′	0.053	0.316	0.25	0.106
O	0.183	0.293	0.16	0.366
βC	0.044	0.000	−2.39	0.088

in adjacent chains displaced to the front and the rear of the plane of the sheet, so that it is appropriate to describe it as a pleated sheet. We suggest that this sheet be called the antiparallel-chain pleated sheet, the original one being called perhaps the polar pleated sheet, inasmuch as all of the carbonyl groups are oriented in the same direction, either to the right or to the left, the imino groups being oppositely directed.

The second way of arranging the chains of type 1 into a sheet involves parallel chains, which also are arranged to form a pleated sheet. Diagrammatic representations of the antiparallel-chain pleated sheet and parallel-chain pleated sheet are shown in figures 4 and 5, and drawings of these structures in figures 6 and 7, respectively.

Atomic coordinates for the two pleated sheets are given in table 2. The

antiparallel-chain pleated sheet, considered as made of L-amino acid residues, has orthorhombic symmetry, its symmetry elements being a twofold screw axis in the direction of the chains, a twofold screw axis normal to this direction and in the plane of the sheet, and a twofold axis normal to the plane of the sheet. The parallel-chain pleated sheet has only a twofold screw axis in the direction of the chains.

In the parallel-chain pleated sheet the oxygen atom lies nearly in the N—H direction. The deviation from the N—H axis would be zero in case that the bond angle C—N—H were equal to 125°. It is likely that this bond angle is about 116°, and that there is accordingly a bend in the hydrogen bond of about 10°. This is not so great as to lead to significant instability.

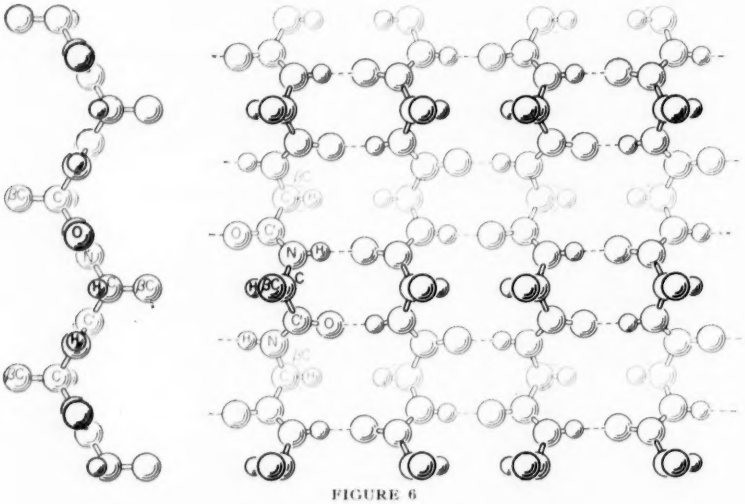

FIGURE 6

Drawing representing the anti-parallel-chain pleated sheet structure.

It is interesting to note that the ability of polypeptide chains with configuration 1 to form lateral hydrogen bonds with either a parallel or an antiparallel adjacent chain strongly suggests that proteins with the β-keratin structure may exist in which there is a randomness in sequence of parallel and antiparallel chains. We plan to discuss this question later, in connection with a comparison of observed and calculated intensities of x-ray reflections for synthetic polypeptides and proteins of the β-keratin type.

It is to be noted that our 5.1-residue helix[1-3] is not represented in table 1, although it satisfies the conditions of containing trans amide groups, and of having all amide groups equivalent. The reason for its failure to appear

as a result of the considerations of the present paper is that it represents unfavorable orientation about both the N—C bond and the C—C′ bond; the orientation about each bond is such as to correspond to a rotation of about 30° for both C—H and C—R from the position of coplanarity with each of the two amide groups adjacent to the α carbon atom. If we accept the assumption expressed above that these orientations represent an instability of about 1 kcal. mole^{-1} relative to the favored orientations, then the 5.1-residue helix would be predicted to be less stable than the 3.7-residue helix by 2 kcal. mole^{-1} per residue. There is no indication from the structures that a corresponding extra stability would be conferred on the 5.1-residue helix by, for example, the presence of N—H···O hydrogen bonds of abnormally great stability, the absence of steric hindrance between side

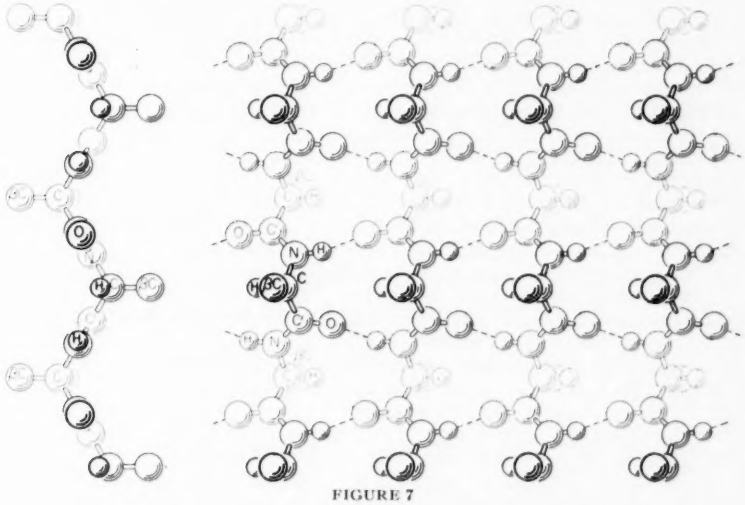

FIGURE 7

Drawing representing the parallel-chain pleated sheet structure.

chains of adjacent turns of the helix which might be present in the 3.7-residue configuration and make it less stable, or any other structural feature—indeed, as was pointed out in an earlier paper,[23] the cylindrical cavity down the center of the 5.1-residue helix would be expected to lead to an extra instability for this configuration by decreasing the intramolecular van der Waals attraction. We thus are led to the conclusion that the 5.1-residue helix is a less stable configuration of polypeptide chains than the 3.7-residue helix, and that it would not be expected to occur very often as an important feature of the structure of proteins. In particular, we are provided with grounds for withdrawing our tentative suggestion[7] that pro-

teins of the α-keratin type undergo transformation to the 5.1 residue configuration on supercontraction. This suggestion was made because the amount of contraction involved in the change from the 3.7-residue helix to the 5.1-residue helix, about 35 per cent, is just that reported to occur in the process of supercontraction. We now think that it is likely that, as suggested by Astbury, the process of supercontraction involves merely a disorientation of α-keratin molecules, without significant change in the configuration of the individual polypeptide chains.

This investigation was aided by grants from The Rockefeller Foundation, The National Foundation for Infantile Paralysis, and The National Institutes of Health, Public Health Service.

* Contribution No. 1629.

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HYBRIDS BETWEEN LYCOPERSICON ESCULENTUM MILL. AND SOLANUM LYCOPERSICOIDES DUN.

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Communicated by Karl Sax, September 14, 1951

The garden tomato (*Lycopersicon esculentum* Mill.) has been successfully hybridized with all other species of *Lycopersicon* that have been grown for experimental purposes.[1, 2] Its crossability relations with the large parent genus, *Solanum*, on the other hand, are largely unknown, although a strong barrier to crossing with certain species might be inferred from the following observations. *Solanum melongena*, *S. muricatum*, *S. nigrum*, *S. tuberosum*, and other species, especially of the tropics, are often grown close to tomatoes, or are found as weeds in tomato fields. Despite this opportunity for crossing, spontaneous hybrids between the tomato and these species have apparently never appeared, and apparently no such hybrid has been produced artificially. The present example of a tomato-nightshade hybrid was encountered in an exploratory survey of the relations between the two genera.

The tomato parent of the hybrids is Pearson, the leading variety of the California tomato industry. It is typical of the standard or normal-type *L. esculentum* in the genetic sense except that it is homozygous for the *sp* gene, which conditions determinate plant habit.

The nightshade parent is *Solanum lycopersicoides* Dun., a perennial native to the south coast of Peru. It has been classed by Bitter[3] in the section *Tuberarium*, the group of *Solanum* species usually considered to be most closely allied to the tomatoes.[4] Even among species of this section, *S. lycopersicoides* shows a unique resemblance, as its name implies, to the tomatoes in such characteristics as its yellow flowers, nature of leaf dissection, and lack of tubers.

Taxonomically *S. lycopersicoides* is a true *Solanum* in having the following features that distinguish it clearly from *Lycopersicon:* its anthers lack sterile tips, dehisce terminally and do not join laterally to each other to form an anther tube.

Plants of *S. lycopersicoides* were acquired[5] as collection No. 30382 of the University of California Botanical Garden Third Expedition to the Andes, 1942. Seeds had been collected near Candarave, Peru (Dept. Tacna, Prov. Tarata). Plants in cultures at Davis resemble very closely the herbarium specimen of the original collection (Spec. No. 695,139 of the University of California Herbarium).

The F_1 hybrids between these two species were obtained with considerably greater ease than those between *L. esculentum* and *L. peruvianum* (L.) Mill.

The parents were crossed in both directions, but since fruits are set erratically on the nightshade parent even after selfing and intraspecific crossing, the tomato was mostly used as the pistillate parent. Following pollinations of the latter kind, small fruits are set containing ovules that develop to an intermediate size. A few small seeds reach maturity. Since these seeds did not germinate, 30-day fruits were opened under sterile conditions, the ovules dissected, and the endosperm-embryo contents placed in sterile cultures of 0.8% agar supplemented with Hoagland and Snyder's solution,[6] minor elements, and 2% dextrose. In these cultures embryos grow slowly at first, many being deformed, but after an initial period of slow growth, they develop very rapidly.

Fifteen embryos were successfully cultured in this fashion, and eight of these have been transferred to soil. Further vegetative growth in the greenhouse and field is not attended with any delay or abnormality; in fact, under field conditions the hybrids grow with much greater vigor than either parent.

The two parent species differ in a great number of readily distinguished characters. Comparisons between the parents and F_1 hybrids for 25 of these traits have been made. Each sample consisted of 20 or more measurements. Only three characters of *L. esculentum*—day-length response, fragrance of flower and relative calyx segment length—are dominant in the hybrids. Of the remaining 22, five of *S. lycopersicoides* are completely dominant in the hybrids, the other 17 being intermediate but usually showing greater resemblance to *S. lycopersicoides*. In respect to the key characters of the anther the hybrid would very likely be annotated as a Solanum; in respect to general appearance the hybrid seems like a magnified *S. lycopersicoides*. Considering the morphology, the subsequently described sterility, and chromosome behavior of these plants, their hybrid origin can scarcely be doubted.

The fertility of the hybrids is extremely low. Of the pollen grains formed, only 0.5% are stained by acetocarmine, and many of these appear otherwise defective. Ovule fertility as assayed in self-pollinations and backcrosses to both parents, though not yet completely analyzed, is at least as low as the pollen fertility. If *L. esculentum* is used as the pistillate parent in backcrosses, occasional ovules develop, and preliminary embryo cultures have yielded one seedling. It is suspected that with perseverance a backcross progeny might be grown, and that characters might be transferred from *S. lycopersicoides* to *L. esculentum*.

Chromosome behavior can explain at least part of the very low fertility of the F_1 hybrids. Both parents possess 12 pairs of chromosomes, those of *S. lycopersicoides* appearing larger at metaphsase than those of *L. esculentum*. The cytology of two hybrid seedlings was studied, each revealing the expected 24 chromosomes. In 79 figures at metaphase of the first meiotic

division a mean of 14.8 of the 24 chromosomes was paired, mostly as bivalents, only a few trivalents and quadrivalents being observed. The number of paired chromosomes varied from 8 to 24. The unpaired chromosomes lag and usually fail to enter telophase nuclei. A mean of 9.6 chiasmata was observed in 30 pollen-mother cells at diakinesis. Occasional figures exhibit chromosome bridges.

Considering this cytological picture and the nearly perfect pairing of chromosomes in all hybrids between species of *Lycopersicon*, the chromosomes of *L. esculentum* seem to be closer in homology to those of other tomato species than to those of *S. lycopersicoides*.

The relatively low degree of chromosome pairing in the hybrid suggests that some degree of fertility might be restored in amphidiploids. This expectation has been confirmed to the extent that tetraploids obtained from colchicine treatment exhibit 67% pollen fertility and nearly complete pairing of chromosomes, mostly as bivalents.

The feasibility of obtaining *Lycopersicon-Solanum* hybrids opens a new and interesting opportunity for further research in the genetics and breeding of the tomato. In addition to the many known features of the tomato that favor it for cytogenetic research, a genetic link with the *Solanum* species is now available that provides an avenue for exchange of genes between the two genera. Too little is known of the traits of *S. lycopersicoides* to permit any evaluation of it as a parent for practical breeding purposes; in fact, it is not even known whether its fruits contain any of the poisonous alkaloids that are often encountered in fruits of other nightshade species. At least, it is known to be immune to, or tolerant of, tobacco mosaic, for, despite frequent contact with infected tomato plants during two years of clonal culture, no symptoms have appeared.

It might be appropriate to consider here the bearing of certain facts about the hybrids upon the relations between the parent genera, although an extensive consideration of phylogeny here would be premature. A reasonably close relationship between the parents of the present hybrids must be assumed because the latter were obtained with relative ease and because they have great vegetative vigor. Although the hybrids are highly sterile, backcrosses to the tomato parent and possibly to both parents can succeed, thereby providing a medium of exchange of genes between the two parents. Such features are encountered in many known hybrids between species of the same genus—for example, *Nicotiana*. In fact, most interspecific crosses within plant genera fail, and many that succeed yield hybrids of low viability and of even greater sterility than the present examples. The latter thus behave genetically more like hybrids between species of the same genus rather than of different genera, and one is led to question the separation of *Lycopersicon* and *Solanum* into different genera.

Summary.—F_1 hybrids have been obtained between *Lycopersicon*

esculentum and *Solanum lycopersicoides* by resort to embryo culture. The hybrids are more vigorous vegetatively than either parent and in the majority of morphological characters resemble the nightshade parent more closely than the tomato parent. Pollen and ovule fertility are very low, but it is possible to secure progeny from the backcross to the tomato parent. The hybrids possess 24 chromosomes, the same number present in both parents. Slightly more than half of the chromosomes pair in the diploid hybrids, but in colchicine-induced amphidiploids nearly all of the chromosomes pair, chiefly as bivalents and pollen fertility is greatly increased.

* This study was supported in part by a grant from the John Simon Guggenheim Memorial Foundation.

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A MAJORANT METHOD FOR NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Communicated by T. Y. Thomas, September 18, 1951

1. Let D be a finite simply connected domain in the (x, y)-plane bounded by a smooth curve C. Consider the non-linear partial differential equation

$$\Delta u = P(x, y)u + \sum_{\nu=2}^{\infty} a_\nu(x, y)u^\nu, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

where $P(x, y)$ is a positive continuous function in D, and the coefficients $a_\nu(x, y)$ are arbitrary continuous functions in D satisfying the inequalities

$$|a_\nu(x, y)| \leqslant K\frac{l^\nu}{\nu!}, \qquad K > 0, l > 0 \text{ constants.} \tag{2}$$

This last inequality guarantees the convergence of the series on the right-hand side of (1) for arbitrary values of u.

We prescribe the boundary values $\mu(s)$ of $u(x, y)$ on C ($0 \leqslant s \leqslant L$, s = length parameter on C) and ask for a solution of (1) assuming these bound-

ary values on C. The value $u(x, y)$ of the solution at a given point $(x, y) \in D$ is a functional of the boundary values $\mu(s)$ and it is natural to try to develop the solution $u(x, y)$ into a functional power series of $\mu(s)$.[5] For sake of simplicity, we reduce this development immediately to an ordinary power series, as follows. Let $u(x, y; t)$ be that solution of (1) which has on C the boundary values $t\mu(s)$ and let us set up:

$$u(x, y; t) = \sum_{\nu=1}^{\infty} u_\nu(x, y)t^\nu \tag{3}$$

We assume the following boundary values for the coefficients $u_\nu(x, y)$ of this development:

$$u_1(x, y) = \mu(s), \qquad u_\nu(x, y) = 0 \qquad \text{for } \nu > 1, \ (x, y) \in C. \tag{4}$$

2. We show at first that if a development (3) exists at all it is uniquely determined by the differential equation and the boundary conditions (4). In fact, introducing (3) into (1) and comparing equal powers of t, we obtain

$$\begin{aligned} \Delta u_1 &= P(x, y)u_1, & u_1 &= \mu(s) \text{ on } C, \\ \Delta u_2 &= P(x, y)u_2 + a_2(x, y)u_1^2, & u_2 &= 0 \text{ on } C, \\ &\vdots & & \\ \Delta u_n &= P(x, y)u_n + E_n(a_2, \ldots a_n, u_1, \ldots u_{n-1}), & u_n &= 0 \text{ on } C. \end{aligned} \tag{5}$$

Here E_n is a well-defined polynomial, linear, homogeneous in the a_ν's and of degree n in the u_ν's with positive coefficients which is easily computed for any given positive integer n.

The determination of the coefficients $u_\nu(x, y)$ is thus reduced to the well-known theory of the linear partial differential equation:[1-4]

$$\Delta u = P(x, y)u, \qquad P(x, y) > 0 \text{ in } D. \tag{6}$$

Let $G(x, y; \xi, \eta)$ be the Green's function for this equation with respect to the domain D. We may then solve the inhomogeneous linear differential equation for $u_n (n > 1)$ in the form:

$$u_n(x, y) = -\iint_D G(x, y; \xi, \eta)E_n(a_2(\xi, \eta), \ldots, u_{n-1}(\xi, \eta))d\xi\, d\eta. \tag{7}$$

Since $u_1(x, y)$ is uniquely determined and can be easily expressed in terms of Green's function also:

$$u_1(x, y) = \int_C \mu(s) \frac{\partial}{\partial \nu} G(x, y; \xi, \eta)\, ds, \ \nu = \text{interior normal}, \tag{8}$$

we can successively compute all coefficients of the development (3). The treatment of the linear partial differential equation (6) may be reduced to the theory of orthogonal particular solutions and to the theory of the kernel

function. From a numerical point of view, the development (3) together with formulas (7) and (8) seems, therefore, to be quite convenient.

3. The decisive problem in our investigation is now the convergence of the series (3). We will study it by means of a majorant procedure based upon the dependence of the Green's function $G(x, y; \xi, \eta)$ upon the coefficient $P(x, y)$ and upon the domain D. We remark at first that Green's function is positive in D and that, therefore, $\frac{\partial G}{\partial \nu} > 0$ on C. Let

$$\max_{0 \leq s \leq L} |\mu(s)| = A. \tag{9}$$

We replace our previous boundary value problem by the following: To find a solution for the partial differential equation

$$\Delta u^* = P(x, y)u^* - K\sum_{\nu=2}^{\infty} \frac{(lu^*)^\nu}{\nu!} \tag{10}$$

which has on C the constant boundary value A. Let $u^*(x, y; t) = \sum_{\nu=2}^{\infty} u_\nu^*(x, y)t^\nu$ be the solution of (10) with the boundary value tA on C, and let E_n^* have an analogous definition to E_n. In view of (8), the coefficient $u_1^*(x, y)$ in the new problem will be positive and at each point larger than the absolute value of the previous $u_1(x, y)$. Since each new $-E_n^*$ will be larger than the absolute value of the corresponding E_n and since Green's function is positive, we will have

$$u_n^*(x, y) \geq |u_n(x, y)|. \tag{11}$$

The series (3) will therefore, converge absolutely wherever the corresponding series for $u^*(x, y; t)$ does converge. We may, therefore, restrict ourselves to the convergence proof for this new series.

Let

$$\min_{(x, y) \in D} P(x, y) = m. \tag{12}$$

It is well known that Green's function for a fixed domain D decreases everywhere if the coefficient $P(x, y)$ of the equation (6) is replaced by one which is larger everywhere in D.[3] Hence, let $G_m(x, y; \xi, \eta)$ be the Green's function of D with respect to the partial differential equation

$$\Delta u = mu; \tag{13}$$

we have then the inequality

$$G(x, y; \xi, \eta) \leq G_m(x, y; \xi, \eta), \tag{14}$$

and since both Green's functions vanish on C, we also have

$$\frac{\partial G}{\partial \nu} \leqslant \frac{\partial G_m}{\partial \nu} \qquad \text{on } C. \tag{15}$$

Let us replace the partial differential equation (10) by the simpler equation

$$\Delta v = mv - K \sum_{\nu=2}^{\infty} \frac{(lv)^\nu}{\nu!} \tag{16}$$

with the same boundary condition $v = A$ on C. Defining in analogy to (3)

$$v(x, y; t) = \sum_{\nu=1}^{\infty} v_\nu(x, y) t^\nu, \tag{17}$$

we easily conclude from (7), (8), (14) and (15) that

$$v_\nu(x, y) \geqslant u_\nu^*(x, y) \geqslant |u_\nu(x, y)|. \tag{18}$$

Let, finally, Γ_R be a circle of radius R which contains the domain D. Consider the radially symmetric solution of the linear partial differential equation

$$\Delta w = mw, \qquad w = w(r), \tag{19}$$

which is normalized in such a way that its minimum on C is exactly one. Let $\lambda(C)$ be the value of $w(R)$; this number depends on the geometry of C and can easily be expressed in terms of Bessel functions.

We now pose the following boundary value problem: To find a solution $U(x, y)$ of the partial differential equation (16) in Γ_R which has on the circumference the constant boundary value $A\lambda(C)$. We define again the power series

$$U(x, y; t) = \sum_{\nu=1}^{\infty} U_\nu(x, y) t^\nu. \tag{20}$$

We remark that $U_1(x, y)$ is defined by the equation:

$$\Delta U_1 = mU_1, \qquad U_1 = A\lambda(C) \qquad \text{on } \Gamma_R. \tag{21}$$

Comparing this equation with (19), we find

$$U_1(x, y) = Aw(r) \tag{22}$$

and since we have

$$\Delta v_1 = mv_1 \qquad v_1(x, y) = A \leqslant U_1(x, y) \text{ on } C, \tag{23}$$

we derive by the maximum principle

$$v_1(x, y) \leqslant U_1(x, y) \qquad \text{in } D. \tag{24}$$

We denote Green's function for the circular domain and the differential

equation (13) by $g(x, y; \xi, \eta)$. Since Green's function increases monotonically with the domain, we have the inequality

$$g(x, y; \xi, \eta) \geq G_m(x, y, \xi, \eta), \qquad (x, y) \in D. \tag{25}$$

Since the U_ν as well as the v_ν are obtained by recursion formulas (7) based on the Green's functions, we have by (24) and (25):

$$U_\nu(x, y) \geq v_\nu(x, y) \geq |u_\nu(x, y)|. \tag{26}$$

Thus, the convergence of the series (3) will be ensured, if we can prove the convergence of the corresponding series in the particular boundary value problem for $U(x, y)$. But in this case, $U(x, y)$ will only depend on the radius r to the center of the circle Γ_R. We may reduce the whole question to a problem in ordinary differential equations. In fact, $U(x, y) = U(r)$ satisfies the ordinary differential equation

$$\frac{d^2}{dr^2} U(r) + \frac{1}{r} \frac{d}{dr} U(r) = mU - K[e^{lU} - 1 - lU] \tag{27}$$

with the boundary conditions $U(0) =$ finite, $U(R) = A\lambda(C)$. It is easily seen that the solution of (27) depends analytically upon $U(R)$ if this value is small enough. Hence, the series development (3) will converge for $\max |\mu| = A$ sufficiently small. The existence theorem for the boundary value problem with respect to the non-linear partial differential equation (1) has been reduced to the study of a boundary value problem for the ordinary differential equation (27).

It should be pointed out that in the theory of partial differential equations in a given domain D the majorant method is based on a comparison of the Green's functions for different domains and different differential equations. It is a procedure in the large in contradistinction to the Cauchy-Kowalevsky method of majorant power series which guarantees solutions only locally and is, therefore, not applicable in boundary value problems.

The method of majorants can readily be extended to other problems in non-linear partial differential equations of elliptic type. It may be applied to the case of more than two independent variables and may also be extended to the case of boundary value problems of the second and the third type. In the latter problems, the Neumann's and the Robin's functions will play analogous roles to the Green's function in the above development.

In order to study the breakdown of the series development (3) and the uniqueness of the solutions for given boundary values, one has to study the analytic function (3) in dependence of its argument t in the complex plane and to continue it analytically beyond its singular points on the real axis, if possible.

* Work done under contract with the Office of Naval Research, NR-043-068.

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ON FINITE GROUPS WITH TWO INDEPENDENT GENERATORS. III. EXPONENTIAL SUBSTITUTIONS

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Communicated September 18, 1951

The two previous notes of this series have appeared in earlier issues of these PROCEEDINGS.[1] This one is a direct continuation.

16. Conjugates of a Given Linear Substitution.—Let

$$\varphi(x) \equiv \beta x \pmod{n}, \qquad (\beta, n) = 1 \tag{16.1}$$

be a given linear substitution,[2] not reducing to the identity: $\beta \not\equiv 1 \pmod{n}$.[3] We deal now with the problem of finding all its conjugates θ.

The range m of θ must be a multiple of the period s of φ, as defined by: $\beta^s \equiv 1 \pmod{n}$, where s has its minimum positive value. We write, as usual, $m = su$. u is one of the unknowns in the determination of θ; it will be the modulus in most of the congruences occurring hereafter.

Another unknown is the θ-parameter g:

$$g \equiv \theta(s)/s \pmod{u} \tag{16.2}$$

We know that g must be prime to u; let λ denote the exponent to which g belongs modulo u, i.e., the minimum positive exponent for which the congruence

$$g^\lambda \equiv 1 \pmod{u}$$

is valid. g has the properties

$$g^n \equiv 1, \; g^{\varphi^t(1)-1} = g^{\beta^t-1} \equiv 1 \pmod{u}$$

(see Theorem XV, Note II). The last relation is obviously implied for all t by its validity for $t = 1$:

$$g^{\beta-1} \equiv 1 \pmod{u}. \tag{16.3}$$

If $\epsilon = (\beta - 1, n)$, we may summarize by saying that

$$g^{\epsilon} \equiv 1 \pmod{u}, \tag{16.4}$$

or that λ must divide ϵ.

The conditions for conjugate special substitutions θ, φ are:

$$\begin{aligned} &\theta(0) \equiv 0 \pmod{m}, \qquad \varphi(0) \equiv 0 \pmod{n}, \\ &\theta^n(x) \equiv x \pmod{m}, \qquad \varphi^m(x) \equiv x \pmod{n}, \end{aligned} \tag{16.5}$$

$$T_1\theta^y = \theta^{\varphi(y)}, \qquad T_1\varphi^y = \varphi^{\varphi(y)}. \tag{16.6}$$

(Cf. Note I, §3.)

Now $\varphi^m(x) \equiv \beta^m x \equiv \beta^{su} x \equiv x \pmod{n}$; thus the second condition (16.5) is met.

$T_1\varphi^y$ is the transformation

$$x' \equiv \varphi^y(x + 1) - \varphi^y(1) \equiv \beta^y(x + 1) - \beta^y \equiv \beta^y x \pmod{n},$$

and $\varphi^{\theta(y)}$ the transformation

$$x' \equiv \beta^{\theta(y)} x \pmod{n}.$$

The second condition (16.6) is therefore equivalent to

$$\theta(y) \equiv y \pmod{s}. \tag{16.6a}$$

Accordingly, we may write

$$\theta(x) \equiv x + s\psi(x) \pmod{m}, \tag{16.7}$$

$\psi(x)$ being an unknown function, whose determination is precisely our goal. Of course, we must have

$$\psi(0) \equiv 0 \pmod{u} \tag{16.8}$$

in order that $\theta(0) \equiv 0 \pmod{m}$.

A further necessary condition is that θ admit the translation T_s; this is equivalent to

$$\psi(x + s) \equiv \psi(x) + \psi(s) \pmod{u}.$$

From (16.2), (16.7),

$$g \equiv 1 + \psi(s) \pmod{u}; \tag{16.9}$$

hence

$$\psi(x + s) \equiv \psi(x) + g - 1 \pmod{u}. \tag{16.10}$$

By induction on M, we infer

$$\psi(x + Ms) \equiv \psi(x) + M(g - 1) \pmod{u}. \qquad (16.11)$$

In order that θ really represent a *substitution* on the range m, we must first require that if $x \equiv y \pmod{m}$, then $\theta(x) \equiv \theta(y) \pmod{m}$; equivalent to this (by (16.7)) is $\psi(x) \equiv \psi(y) \pmod{u}$, which follows from (16.11) when we replace M by Mu. Conversely, $\theta(x) \equiv \theta(y) \pmod{m}$ must imply $x \equiv y \pmod{m}$. Now, in fact, $\theta(x) \equiv \theta(y) \pmod{m}$ implies $\theta(x) \equiv \theta(y) \pmod{s}$, which (by (16.6a)) implies $x \equiv y \pmod{s}$, or $x = y + Ms$. Therefore (by (16.7)): $y + Ms + s\psi(y + Ms) \equiv y + s\psi(y) \pmod{m}$, $M + \psi(y + Ms) \equiv \psi(y) \pmod{u}$, $Mg \equiv 0 \pmod{u}$ (by (16.11)), $M \equiv 0 \pmod{u}$—since g is prime to u, finally $x \equiv y \pmod{m}$, Q.E.D.

It remains only to render valid the first conditions in (16.5) and (16.6) by proper choice of the function ψ.

We have

$$\begin{aligned} \theta^2(x) &\equiv x + s\psi(x) + s\psi(x + s\psi(x)) \\ &\equiv x + s\psi(x) + s\{\psi(x) + (g - 1)\psi(x)\} \\ &\equiv x + s\psi(x)\,(1 + g) \pmod{m}, \end{aligned}$$

and by induction on y we readily establish:

$$\theta^y(x) \equiv x + s\psi(x)\,\frac{g^y - 1}{g - 1} \pmod{m}. \qquad (16.12)$$

If $g = 1$, the fraction $(g^y - 1)/(g - 1)$, being $= 1 + g + g^2 + \ldots + g^{y-1}$, is to be interpreted as equal to y. *This will be a standing convention throughout, whatever letters appear in place of g and y;* e.g., in formula (16.18), the fraction $= x$ if $b = 1$.

Using (16.12), we find the first condition (16.6) to be expressed by the formula

$$\frac{g^y - 1}{g - 1}\,\{\psi(x + 1) - \psi(1)\} \equiv \psi(x)\,\frac{g^{\beta y} - 1}{g - 1} \pmod{m} \qquad (16.13)$$

for all x, y. Now

$$(g^{\beta y} - 1)/(g - 1) = \{(g^{\beta y} - 1)/(g^\beta - 1)\} \cdot \{(g^\beta - 1)/(g - 1)\},$$

and

$$\begin{aligned} (g^{\beta y} - 1)/(g^\beta - 1) &= 1 + g^\beta + g^{2\beta} + \ldots + g^{(y-1)\beta} \\ &\equiv 1 + g + g^2 + \ldots + g^{y-1} \pmod{u} \\ &= (g^y - 1)/(g - 1), \qquad (16.14) \end{aligned}$$

where we use the fact that $g^\beta \equiv g \pmod{u}$, by (16.3).

Hence (16.13) becomes

$$\frac{g^y - 1}{g - 1}\{\psi(x + 1) - \psi(1)\} \equiv \frac{g^y - 1}{g - 1} \cdot \frac{g^\beta - 1}{g - 1}\psi(x) \pmod{u},$$

which will hold for all y if and only if it holds for $y = 1$:

$$\psi(x + 1) - \psi(1) \equiv \frac{g^\beta - 1}{g - 1}\psi(x) \pmod{u}. \tag{16.15}$$

Defining

$$a \equiv \psi(1), \qquad b \equiv \frac{g^\beta - 1}{g - 1} \pmod{u}, \tag{16.16}$$

we write (16.15) in the form

$$\psi(x + 1) \equiv a + b\psi(x) \pmod{u}. \tag{16.17}$$

This gives

$$\psi(x + 1) - \psi(x) \equiv b\{\psi(x) - \psi(x - 1)\} \pmod{u},$$

or

$$\Delta\psi(x) \equiv b\Delta\psi(x - 1), \text{ where } \Delta\psi(x) = \psi(x + 1) - \psi(x).$$

Consequently, $\Delta\psi(x) \equiv ab^x$ $(x \geq 0)$. Replacing x by y and summing from $y = 0$ to $y = x - 1$, with account taken of (16.8), we get

$$\psi(x) \equiv a\frac{b^x - 1}{b - 1} \pmod{u} \qquad (x \geqq 0). \tag{16.18}$$

(For $b = 1$, the fraction is to be interpreted as equal to x.)

Formula (16.9) imposes the condition

$$a\frac{b^s - 1}{b - 1} \equiv g - 1 \pmod{u}, \tag{16.19}$$

and with use of the fact that $(b - 1)(g - 1) \equiv 0 \pmod{u}$ (by (16.16) and (16.3)), we easily see that this secures the requirement (16.10). Formula (16.11) may be used to extend the definition of $\psi(x)$ to negative values of x.

It only remains now to take account of the first condition (16.5); by (16.12), (16.18), this leads to

$$a\frac{b^x - 1}{b - 1} \cdot \frac{g^n - 1}{g - 1} \equiv 0 \pmod{u},$$

which is valid for all x if valid for $x = 1$, so giving us our final condition

$$a\frac{g^n - 1}{g - 1} \equiv 0 \pmod{u}. \tag{16.20}$$

Since $g^\lambda \equiv 1 \pmod{u}$ and λ divides both $\beta - 1$ and n (cf. 16.4)), we can transform (16.16) and (16.20) as follows:

$$b \equiv 1 + \frac{\beta - 1}{\lambda} \frac{g^\lambda - 1}{g - 1} \pmod{u} \tag{16.21}$$

$$a \frac{n}{\lambda} \frac{g^\lambda - 1}{g - 1} \equiv 0 \pmod{u}. \tag{16.22}$$

This is done by a procedure like that which gave us (16.14).

In summary, to find in the most general way a conjugate substitution θ to the linear one φ defined in (16.1), we must first select arbitrarily a value of u to give us the range $m = su$ of θ, then an integer g, prime to u, which obeys (16.4), then define b by (16.21), then, if possible, determine a so as to obey simultaneously the congruences (16.19), (16.22). θ will then be defined by (16.7) with $\psi(x)$ given by (16.18):

$$\theta(x) \equiv x + sa \frac{b^x - 1}{b - 1} \pmod{m}. \tag{16.23}$$

By (16.12), we see that the powers of θ are given by the formula

$$\theta^y(x) \equiv x + sa \frac{g^y - 1}{g - 1} \frac{b^x - 1}{b - 1} \pmod{m} \tag{16.23a}$$

if $g \not\equiv 1 \pmod{u}$, and by the formula

$$\theta^y(x) \equiv x + say \frac{b^x - 1}{b - 1} \pmod{m} \tag{16.23b}$$

if $g \equiv 1 \pmod{u}$.

From this we infer that, if $g \not\equiv 1 \pmod{u}$, the period r of θ is the least positive integral exponent such that

$$g^r \equiv 1 \pmod{u(g - 1)/(a, u)}, \tag{16.23c}$$

while, if $g \equiv 1 \pmod{u}$,

$$r = u/(a, u). \tag{16.23d}$$

$r = n$ is the necessary and sufficient condition that the given φ be the derivative of θ: $\varphi = \theta_1$.

Since by Theorem XVI in Note II, §9,

$$g^r \equiv 1 \pmod{u}, \tag{16.23e}$$

one may observe, by noting the definition of λ following (16.2), that $\lambda | r$.

The Case $g \equiv 1$ (Mod u).—It is convenient to abbreviate the system (16.19, 22) to:

$$aB \equiv g - 1, \qquad aC \equiv 0 \pmod{u} \tag{16.24}$$

by writing

$$B \equiv \frac{b^s - 1}{b - 1}, \qquad C \equiv \frac{n}{\lambda}\frac{g^\lambda - 1}{g - 1} \pmod{u}. \tag{16.25}$$

We define

$$\Delta = (B, C, u), \qquad D = (C, u). \tag{16.26}$$

Obviously, $\Delta = (B, D)$.

The determination of a is always possible if we select $g \equiv 1 \pmod{u}$, which is in accord with (16.4).

For the system (16.24) then becomes

$$aB \equiv 0, \qquad aC \equiv 0 \pmod{u}, \tag{16.27}$$

where, since $\lambda = 1$ and $b \equiv \beta \pmod{u}$ (see (16.16) and the convention stated just after (16.12)), we have

$$B \equiv \frac{\beta^s - 1}{\beta - 1}, \qquad C \equiv n \pmod{u}. \tag{16.28}$$

The solution for a of (16.27) is:

$$a \equiv 0 \pmod{u/\Delta}. \tag{16.29}$$

This gives us precisely Δ values of a that are incongruent modulo u: $a = M(u/\Delta)$, $M = 0, 1, 2, \ldots, \Delta - 1$. The substitution θ (16.23) corresponding to a general value of M is the Mth power of θ', where θ' corresponds to $M = 1$.

We may observe that the period r of θ is equal to $\Delta/(M, \Delta)$. Obviously, (16.26, 28), $\Delta | n$, therefore $r | n$, as required by the conditions for conjugacy of θ, φ. The derivative of θ is: $\theta_1(x) \equiv \beta x \pmod{r}$. θ is linear in case $\beta \equiv 1 \pmod{r}$, and is otherwise non-linear, therefore of type 3 (cf. the remarks associated with formula (16.34)).

The Case $g \not\equiv 1$ (Mod u).—For a general value of g, a brief study of the system (16.24) gives the following results.

Let the integer P be determined by the congruence

$$P(B/\Delta) \equiv 1 \pmod{D/\Delta}. \tag{16.30}$$

This is always possible, since $\Delta = (B, D)$, and therefore B/Δ, D/Δ are relatively prime.

Then

$$a \equiv P(g - 1)/\Delta \pmod{u/\Delta}, \tag{16.31}$$

provided that (i) $P(g-1)/\Delta$ is an *integer* which, (ii) when substituted for a, obeys the congruences

$$aB \equiv g-1, \qquad aC \equiv 0 \pmod{u}. \tag{16.32}$$

Otherwise the system of congruences (16.19), (16.22) for a is inconsistent, and there is no substitution θ corresponding to the values selected for u and g. In the case of consistency, we have Δ values of a in arithmetic progression:

$$a = P(g-1)/\Delta + M(u/\Delta), \qquad M = 0, 1, 2, \ldots, \Delta - 1. \tag{16.33}$$

For the first and second differences of the function $\theta(x)$ in (16.23), we have

$$\Delta\theta(x) \equiv 1 + sab^x, \qquad \Delta^2\theta(x) \equiv sab^x(b-1) \pmod{m = su}.$$

It follows that the criterion for the reduction of $\theta(x)$ to a *linear* substitution: $\theta(x) \equiv (1+sa)x \pmod{m}$, is

$$a(b-1) \equiv 0 \pmod{u}. \tag{16.34}$$

Provided that $a(b-1) \not\equiv 0 \pmod{u}$, the substitution defined in (16.23) will be called *exponential*. Since then θ is a conjugate of linear, while not itself linear, it is clear that exponential substitutions are precisely those of type 3. (Cf. Theorem X, Note I and Theorem XXIII, Note II.)

We may also remark that the condition that two exponential substitutions—(16.23) and the same with a', b' in place of a, b—be identical is:

$$a \equiv a', \qquad ab \equiv a'b' \pmod{u}, \tag{16.35}$$

so that exponential substitutions which look different may really be the same. E.g., for all x,

$$x + 4(2^x - 1) \equiv x + 2(1-(-1)^x) \pmod{12},$$
$$x + 1 - (-1)^x \equiv x + 3^x - 1 \pmod{8}.$$

The $(q+1)$th difference of $\theta(x)$, q any positive integer, is

$$\Delta^{q+1}\theta(x) \equiv sab^x(b-1)^q \pmod{m = su};$$

hence if there is a positive integer q such that

$$a(b-1)^q \equiv 0 \pmod{u}, \tag{16.36}$$

the function $\theta(x)$ is congruent mod m to a polynomial of degree $\leqq q$, and exactly of degree q if q is the *minimum* positive integral exponent for which (16.36) is valid. E.g., the type 3 substitution $\theta = (0)(2)(4)(6)(1357)$ is represented exponentially by the formula

$$\theta(x) \equiv x + 1 - (-1)^x \pmod{8}, \tag{16.37}$$

subsumed under the general formula (16.23) by writing $s = 2, u = 4, a = 1, b = -1$. Since here

$$a(b-1)^2 = 4 \equiv 0 \pmod{u},$$

whereas $a(b-1) = -2 \not\equiv 0 \pmod{u}$, $\theta(x)$ must really be quadratic; in fact one easily verifies:

$$\theta(x) \equiv x + 2x^2 \pmod{8}. \tag{16.38}$$

17. Examples.—(*a*) we consider the problem of finding all the conjugates of

$$\varphi(x) \equiv -x \pmod{n}, \tag{17.1}$$

the range n being arbitrary.

We have $\beta = -1, s = 2, \epsilon = (2, n)$.

If n is *odd*, we must have $\epsilon = 1$; therefore $g \equiv 1 \pmod{u}$. This gives us $b \equiv \beta \equiv -1, B \equiv 0, C \equiv n \pmod{u}$. $\Delta = (B, C, u) = (n, u)$. $a \equiv 0 \pmod{u/\Delta}$. In summary, if n is odd, all the conjugates of (17.1) are given by the formula

$$\theta(x) \equiv x + M(u/\Delta)(1 - (-1)^x) \pmod{m = 2u} \tag{17.2}$$

where $M = 0, 1, 2, \ldots, \Delta - 1$. These are the powers of the particular example θ' corresponding to $M = 1$. θ' may be described as the substitution which keeps the even numerals fixed and advances each odd one cyclically by m/Δ, so that each cycle of θ' contains either Δ equally spaced odd numerals or one even numeral. θ is linear if and only if $2M(u/\Delta) \equiv 0 \pmod{u}$, $2M \equiv 0 \pmod{\Delta}$, $M \equiv 0 \pmod{\Delta}$ (since Δ is odd); thus, among the powers of θ' only the identity is linear. If u is prime to n, then $\Delta = 1$, and (17.2) gives only the identity on the range $2u$.

If n is *even*, $\epsilon = 2$; therefore either $g \equiv 1$ or $g^2 \equiv 1, g \not\equiv 1 \pmod{u}$. By choosing $g \equiv 1$, we obtain again the type (17.2). The discussion of the case $g^2 \equiv 1, g \not\equiv 1$ can be carried through in each particular instance according to the plan described in the preceding section. E.g., for $\varphi(x) \equiv -x \pmod{4}$, we find as particular conjugates the substitutions θ, θ^{-1} of example (*b*).

(*b*) Conjugates of

$$\varphi(x) \equiv 3x \pmod{8}. \tag{17.3}$$

We have $\beta = 3, s = 2, m = 2u, \epsilon = (\beta - 1, n) = (2, 8) = 2, g^2 \equiv 1 \pmod{u}$. To obtain a particular example, let us take $u = 8, g = 3$. Then $m = 16, \lambda = 2, b \equiv 5, B \equiv 6, C \equiv 0$ (all mod 8). $\Delta = (6, 0, 8) = 2$, $D = (0, 8) = 8$. $3P \equiv 1 \pmod{4}$, $P \equiv 3 \pmod{4}$, $a \equiv 3 \cdot 2/2 \equiv 3 \pmod{4} \equiv 3,7 \pmod{8}$. Choosing $a \equiv 3 \pmod{8}$, we have

$$\theta(x) \equiv x + \frac{3}{2}(5^x - 1) \pmod{16}. \tag{17.4}$$

Expressed in cycles: $\theta = (0)\ (8)\ (1, 7, 9, 15)\ (3, 13, 11, 5)\ (2, 6)\ (4, 12)\ (10, 14)$. The choice $a \equiv 7 \pmod 8$ gives θ^{-1}. We observe that the basic value $a = P(g-1)/d = 3$ is an integer which obeys $Ba \equiv g - 1$, $Ca \equiv 0 \pmod u$, as required.

The *linear* conjugates θ of φ: $\theta(x) \equiv \alpha x \pmod m$ are precisely those of period 2 on an even range: $\alpha^2 \equiv 1 \pmod m$, $m = 2u$. (Cf. Theorem XXVI, Note II.)

18. Antiderivatives of a Given Linear Substitution.—A conjugate θ of a given substitution φ does not necessarily have φ for its derivative. We can only say: $\varphi(x) \equiv \theta_1(x) \pmod r$, where r, the period of θ and the range of θ_1, is a *divisor* of n, the range of φ. As has been stated, in order that $\varphi = \theta_1$, r must be *equal* to n.

The problem of antiderivatives—*given* φ, *find* θ *such that* $\theta_1 = \varphi$—is important because from a given substitution it enables us to find one of the next higher type.

We consider here the problem of the antiderivatives of a given linear substitution (16.1). The following is the procedure leading to the desired result.

Let φ have the period s, i.e., let β belong to the exponent s, modulo n. Then $m = su$. Choose u arbitrarily, then g prime to u so that $g^\epsilon \equiv 1 \pmod u$, where $\epsilon = (\beta - 1, n)$. Suppose that g belongs to the exponent λ, modulo u; then $\lambda | \epsilon$, so that $\lambda | n$. We calculate:

$$G \equiv \frac{g^\lambda - 1}{g - 1}, \qquad b \equiv 1 + \frac{\beta - 1}{\lambda} G, \qquad B \equiv \frac{b^s - 1}{b - 1} \pmod u. \tag{18.1}$$

Let

$$\delta = (G, u), \qquad u = d\delta. \tag{18.2}$$

Then to allow us to proceed further,

$$\rho = n/\lambda \text{ must divide } (B, d). \tag{18.3}$$

Suppose that this condition is met, and let the integer R be defined by the congruence

$$B/\rho \equiv R \pmod \delta. \tag{18.4}$$

Since B is determined mod u, B/ρ is determined mod $u/\rho = (d/\rho)\delta$; hence R is indeed determined mod δ, since d/ρ is an integer by the requirement (18.3).

Now find E, the g.c.d. of R and δ; then a requirement for further procedure is that

$$E \text{ divide } (g-1)/d \tag{18.5}$$

(the latter always an integer). If this condition is fulfilled, let the relatively prime integers H, K denote the cofactors of E in R, δ:

$$R = EH, \qquad \delta = EK. \tag{18.6}$$

The congruence for A:

$$AH \equiv (g-1)/Ed \pmod{K} \tag{18.7}$$

will then always have a solution A_0, unique mod K, so that the general value of A is

$$A = A_0 + MK \tag{18.8}$$

with M an arbitrary integer multiplier.

Now M must be selected so that A is prime to ρ: $(A, \rho) = 1$. Only the values of M in the interval $0 \leqq M < \rho E$ will come into consideration, and, indeed, the values $0 \leqq M < \rho$ are decisive for the primality of A to ρ, since $A_0 + MK$ and $A_0 + (M+\rho)K$ are both prime or both not prime relative to ρ.

In case all this is possible, i.e., if $(g-1)/Ed$ is an integer, and if among the solutions A of the congruence (18.7) there are values prime to ρ, then the substitution of these values in the formula

$$a \equiv A \frac{d}{\rho} \pmod{u} \tag{18.9}$$

gives the appropriate values for a. Values of A congruent mod ρEK give values of a that are congruent mod $dEK = d\delta = u$; this is the reason that, as stated, M may be restricted to the values $0 \leqq M < \rho E$.

The final step is to substitute the values determined for a, b in the general formula (16.23) for θ.

The case $g \equiv 1 \pmod{u}$ is of especial simplicity. The only requirement to be met is:

$$(\beta^s - 1)/(\beta - 1) \equiv 0 \pmod{n}. \tag{18.10}$$

We must then choose $u =$ a multiple of n, $n \mid u$, and have

$$b \equiv \beta, \qquad a \equiv A \frac{u}{n} \pmod{u} \tag{18.11}$$

where A is any integer prime to n. Appropriate substitution is then to be made in the formula (16.23) for θ.

Example: Antiderivatives of

$$\varphi(x) \equiv \beta x \pmod{n} \tag{18.12}$$

where $\beta - 1$, as well as β, is prime to n. This hypothesis obviously insures the fulfillment of condition (18.10).

We have $\epsilon = 1$, so that $g \equiv 1 \pmod{u}$. Then

$$\theta(x) \equiv x + A\,\frac{m}{n}\,\frac{\beta^x - 1}{\beta - 1} \pmod{m}, \tag{18.13}$$

where the range m of θ must be chosen equal to a multiple of ns, $ns \mid m$. A is any integer prime to n, which is equal to the period r of $\theta(x)$. If $\theta'(x)$ corresponds to $A = 1$, then (18.13) represents the substitution θ' to the A power, and defines, for $(A, n) = 1$, all the antiderivatives of φ.

When n is a prime p, the condition: $\beta - 1$ prime to p, is equivalent to $\beta \not\equiv 1 \pmod{p}$. With $n = p$, A an arbitrary non-multiple of p, formula (18.13) thus gives us the most general exponential substitution of period p.

The linear substitutions of period p are defined by

$$\theta(x) \equiv \alpha x \pmod{m}, \qquad \alpha^p \equiv 1, \qquad \alpha \not\equiv 1 \pmod{m}.$$

For the existence of an α obeying these conditions, it is necessary and sufficient that $p^2 \mid m$ or that m have a prime factor of the form $Mp + 1$. This is seen from the formula for Euler's φ-function:

$$m = p^a q^b \ldots r^c, \qquad \varphi(m) = p^{a-1} q^{b-1} \ldots r^{c-1} (p - 1)(q - 1) \ldots (r - 1),$$

and from the fact that those residue classes modulo m which consist of integers prime to m form, by multiplication, an abelian group of order $\varphi(m)$. For an abelian group of order N contains an element of period equal to a specified prime p if and only if p divides N.

We thus have a census of *all* special substitutions of prime period p, since such a one must be either linear or exponential (Theorem XXV, Note II).

The preceding discussion leaves unsettled, for the time being, the question of the existence of an antiderivative for an arbitrarily assigned linear substitution—one where the sufficient condition (18.10) is not verified, such as: $\varphi(x) \equiv 3x \pmod{8}$, or $\varphi(x) \equiv 4x \pmod{9}$.

19. Powers of Special Substitutions.—We deal with the question: when is a given power θ^λ of a special substitution θ of period r also special? The answer is: when and only when the rows of index λy in the array of the corresponding primary group $\Gamma = (\theta, \theta_1)$ form a subgroup of Γ. This occurs when and only when $d \mid \theta_1(dx)$ for all x, where $d = (\lambda, r)$. (Cf. Theorem XVII, Note II.) A tantamount condition is: $d \mid \theta_1^t(d)$ for all t, i.e., the cycle of θ_1 that contains d shall consist entirely of multiples of d. The derivative $y = \theta_1'(x)$ of $\theta' = \theta^\lambda$ is then defined by the congruence

$$\frac{\lambda}{d}\,y \equiv \frac{\theta_1(\lambda x)}{d} \pmod{\frac{r}{d}}, \tag{19.1}$$

always solvable uniquely for the integer y since (i) $\theta_1(\lambda x)/d$ is an integer, (ii) λ/d is prime to r/d.

Algorithmically, to apply (19.1), we may divide by d in those cycles of θ_1 which consist of multiples of d (simply omitting the other cycles, in which none of the numerals is divisible by d), and then multiply each numeral by λ', where $(\lambda/d)\lambda' \equiv 1 \pmod{r/d}$, reducing the result to its least non-negative residue mod r/d.

If θ is an exponential substitution, its derivative θ_1 is linear: $\theta_1(x) \equiv \beta x \pmod{r}$, and the condition: $d \mid \theta_1(dx)$ for all x, is obviously fulfilled. We have for the derivative of $\theta' = \theta^\lambda$: $\theta_1'(x) \equiv \beta x \pmod{r/d}$. Hence:

THEOREM XXVII. *Any power θ^λ of an exponential substitution θ is a special substitution, exponential if $\beta \not\equiv 1 \pmod{r/d}$, linear if $\beta \equiv 1 \pmod{r/d}$, where $d = (\lambda, r)$.*

Of course, any power of a linear substitution is linear. But the property of the last theorem is not characteristic of exponential and linear substitutions. E.g., in the case of the following substitution on the range 32:

$$\theta = (0)\ (16)\ (1, 7, 9, 31, 17, 23, 25, 15)\ (3, 29, 27, 5, 19, 13, 11, 21)\ (2, 6, 18, 22)\ (10, 30, 26, 14)\ (4, 12)\ (8, 24)\ (20, 28),$$

every power of θ is special, but θ, having the derivative

$$\theta_1 = (0)\ (2)\ (4)\ (6)\ (1\ 3\ 5\ 7),$$

of type 3, is itself of type 4, neither exponential nor linear.

The necessary and sufficient condition that every power θ^λ of a given special substitution θ be itself special is that, for every divisor d of r, the cycle of θ_1 containing d shall consist entirely of multiples of d. This requirement is met, in particular, if $r = p^a$, p a prime, and p is a fixed numeral of θ_1. In the example just given, $r = 2^3$, and the numeral 2 is fixed in θ_1.

[1] Vol. 37, nos. 9, 10 (September, October, 1951).

[2] The parenthesis notation is employed for greatest common divisor.

[3] The contrary case has already been disposed of in Note II, §13.

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*MAXIMUM PROPERTIES AND INEQUALITIES FOR THE EIGENVALUES OF COMPLETELY CONTINUOUS OPERATORS**

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Communicated by John von Neumann, September 8, 1951

1. Although several extremum properties of the eigenvalues of completely continuous (abbreviated to c. c.) operators in Hilbert space $\mathfrak{H}$ are known, nearly all of them deal with a single operator. In this respect, the

following theorem of J. von Neumann[1] is of particular significance, as it involves two matrices: *Let A, B be two square matrices of order n (with complex coefficients). If $\{\lambda_i\}$, $\{\kappa_i\}$ are the eigenvalues[2] of A^*A, B^*B respectively, then $\sum_{i=1}^{n} \sqrt{\lambda_i \kappa_i}$ is the maximum of $|tr\ (UAVB)|$, when U, V run over all unitary matrices of order n.* In the present note, Theorem 1 contains two maximum properties (4), (5) for any finite number m of c. c. operators in Hilbert space $\mathfrak{H}$. (4) is a generalization of von Neumann's theorem, while (5) is a multiplicative companion of (4). The case $m = 1$ of (5) includes the important inequality of H. Weyl[3] $|\xi_1 \xi_2 \ldots \xi_n|^2 \leq \lambda_1 \lambda_2 \ldots \lambda_n$ $(n = 1, 2, \ldots)$ *between the eigenvalues $\{\xi_i\}$ of a c. c. operator A in $\mathfrak{H}$ and the eigenvalues $\{\lambda_i\}$ of A^*A.* The case $m = 2$ of (5) includes the following theorem of A. Horn[4] concerning two c. c. operators A, B in $\mathfrak{H}$: *If $\{\lambda_i\}$, $\{\kappa_i\}$, $\{\mu_i\}$ are the eigenvalues of A^*A, B^*B and $(AB)^*(AB)$, respectively, then $\mu_1 \mu_2 \ldots \mu_n \leq \lambda_1 \lambda_2 \ldots \lambda_n \kappa_1 \kappa_2 \ldots \kappa_n$ for $n = 1, 2, \ldots$.* The proof of our Theorem 1 is based on this result of Horn and the elementary Lemma 1. Thus, in the case of finite matrices, our proof of (4) differs from von Neumann's more geometric proof.

For any two c. c. operators A, B in $\mathfrak{H}$, inequality (10) in Theorem 2 gives a comparison, essentially different from Horn's, of the eigenvalues of $(AB)^*(AB)$ with those of A^*A and B^*B, while inequality (11) compares the eigenvalues of $(A + B)^*(A + B)$ with those of A^*A and B^*B. From (11) we derive Theorem 3 on asymptotic distribution of the eigenvalues of $(A + B)^*(A + B)$. In the case of linear integral equations with symmetric kernels, both (11) and Theorem 3 are well-known results of Weyl.[5]

As an application of the maximum property (4), we have in Theorem 5 another inequality comparing the eigenvalues of $(A + B)^*(A + B)$ with those of A^*A, B^*B and involving an arbitrary symmetric gauge function. This generalizes a result of von Neumann[1] concerning finite matrices. To derive Theorem 5 from (4), we first prove Theorem 4 concerning a general inequality for symmetric gauge functions. The geometric meaning of Theorem 4 is clear. Analytically it is analogous to a classical theorem of R. F. Muirhead,[6] although the class of functions involved in his theorem is entirely different from the class of symmetric gauge functions.

2. LEMMA 1A. *Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$. If p_{ij} $(1 \leq i, j \leq n)$ are n^2 non-negative numbers such that*

$$\sum_{i=1}^{n} p_{ij} \leq 1 \text{ for all } j, \qquad \sum_{j=1}^{n} p_{ij} \leq 1 \text{ for all } i, \tag{1}$$

then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}\, a_i\, b_j \leq \sum_{i=1}^{n} a_i\, b_i. \tag{2}$$

Proof: Let $c_i = \sum_{j=1}^{n} p_{ij} b_j$ $(1 \leq i \leq n)$. For a fixed positive integer $k \leq n$, let $q_j = \sum_{i=1}^{k} p_{ij}$ $(1 \leq j \leq n)$. Then $0 \leq q_j \leq 1$ and $\sum_{j=1}^{n} q_j \leq k$. Since $b_j \geq b_{j+1} \geq 0$, we have

$$\sum_{j=1}^{n} q_j b_j \leq \sum_{j=1}^{k-1} q_j b_j + \left(k - \sum_{j=1}^{k-1} q_j\right) b_k \leq \sum_{j=1}^{k-1} (b_j - b_k) + k b_k$$

or

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} b_i \qquad (1 \leq k \leq n). \tag{3}$$

By Abel's lemma, $a_i \geq a_{i+1} \geq 0$ and (3) imply $\sum_{i=1}^{n} a_i c_i \leq \sum_{i=1}^{n} a_i b_i$.

We shall use the above lemma in the following form:

LEMMA 1B. *Let $\{a_i\}$, $\{b_i\}$ be two non-increasing infinite sequences of non-negative numbers such that $\sum_{i=1}^{\infty} a_i b_i < +\infty$. If p_{ij} $(i, j = 1, 2, \ldots)$ are non-negative numbers such that $\sum_{i=1}^{\infty} p_{ij} \leq 1$ for all j and $\sum_{j=1}^{\infty} p_{ij} \leq 1$ for all i, then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{ij} a_i b_j \leq \sum_{i=1}^{\infty} a_i b_i$.*

THEOREM 1. *Let $A_1, A_2, \ldots, A_m$ be m (≥ 1) c. c. operators in Hilbert space $\mathfrak{H}$. For each $j = 1, 2, \ldots, m$, let $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{ji}, \ldots$ be the eigenvalues[2] of $A_j^* A_j$. Then for any positive integer n:*

$$\text{Max} \left| \sum_{i=1}^{n} (U_1 A_1 \ldots U_m A_m x_i, x_i) \right| = \sum_{i=1}^{n} (\lambda_{1i} \lambda_{2i} \ldots \lambda_{mi})^{1/2}, \tag{4}$$

$$\text{Max} \left| \underset{1 \leq i, k \leq n}{\text{Det}} (U_1 A_1 \ldots U_m A_m x_i, x_k) \right|^2 = \prod_{i=1}^{n} (\lambda_{1i} \lambda_{2i} \ldots \lambda_{mi}); \tag{5}$$

where, for both maxima, $U_1, U_2, \ldots, U_m$ independently run over all unitary operators and $\{x_1, x_2, \ldots, x_n\}$ runs over all sets of n orthonormal elements in $\mathfrak{H}$.

Proof: For each $j = 1, 2, \ldots, m$, let H_j be the non-negative square root of $A_j^* A_j$. Then H_j is also c. c. and its eigenvalues are $\{\sqrt{\lambda_{ji}}\}$. Let y_{ji} $(1 \leq i \leq n)$ be n orthonormal elements in $\mathfrak{H}$ such that $H_j y_{ji} = \sqrt{\lambda_{ji}} y_{ji}$ $(1 \leq i \leq n)$. Let $A_j = W_j H_j$ be the polar factorization of von Neumann.[7] Here W_j is a partially isometric operator and its initial set is the closed linear manifold determined by the range of H_j. We have then $W_j^* A_j = H_j$. Since W_j^* is also partially isometric and its initial set is the closed linear manifold determined by the range of A_j, we can find a unitary operator V_j in $\mathfrak{H}$ such that $V_j A_j y_{ji} = W_j^* A_j y_{ji} = H_j y_{ji}$ $(1 \leq i \leq n)$. Let Z_j be a unitary operator in $\mathfrak{H}$ such that $Z_j y_{ji} = y_{j-1,i}$ $(1 \leq i \leq n)$, where we put $y_{0i} = y_{mi}$. Let now

$U_j = Z_j V_j$ $(1 \leq j \leq m)$. Then $U_1 A_1 \ldots U_m A_m y_{mi} = (\lambda_{1i}\lambda_{2i} \ldots \lambda_{mi})^{1/2} y_{mi}$ $(1 \leq i \leq n)$. Hence, for the particular choice of $U_j = Z_j V_j$ $(1 \leq j \leq m)$ and $x_i = y_{mi}$ $(1 \leq i \leq n)$, the relations

$$\left| \sum_{i=1}^{n} (U_1 A_1 \ldots U_m A_m x_i, x_i) \right| \leq \sum_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \ldots \lambda_{mi})^{1/2} \tag{6}$$

$$\left| \operatorname*{Det}_{1 \leq i, k \leq n} (U_1 A_1 \ldots U_m A_m x_i, x_k) \right|^2 \leq \prod_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \ldots \lambda_{mi}) \tag{7}$$

are satisfied and actually become equalities.

It remains to show that inequalities (6), (7) hold for any m unitary operators $U_1, \ldots, U_m$ and any n orthonormal elements $x_1, \ldots, x_n$ in $\mathfrak{H}$. We first prove the case $m = 1$ of (6):

$$\left| \sum_{i=1}^{n} (U_1 A_1 x_i, x_i) \right| \leq \sum_{i=1}^{n} \sqrt{\lambda_{1i}}. \tag{8}$$

Consider the polar factorization of $A_1 = W_1 H_1$. Let $\{y_{1k}\}$ be a complete orthonormal set such that $H_1 y_{1k} = \sqrt{\lambda_{1k}}\, y_{1k}$ $(k = 1, 2, 3, \ldots)$. Then, since $(U_1 A_1 x_i, x_i) = (H_1 x_i, W_1^* U_1^* x_i)$, we have

$$\sum_{i=1}^{n} (U_1 A_1 x_i, x_i) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \sqrt{\lambda_{1k}}\, (x_i, y_{1k})(y_{1k}, W_1^* U_1^* x_i)$$

and therefore

$$2 \left| \sum_{i=1}^{n} (U_1 A_1 x_i, x_i) \right| \leq \sum_{i=1}^{n} \sum_{k=1}^{\infty} \sqrt{\lambda_{1k}} \left| (x_i, y_{1k}) \right|^2 + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \sqrt{\lambda_{1k}} \left| (W_1 y_{1k}, U_1^* x_i) \right|^2.$$

By Lemma 1B, each of the last two double sums is $\leq \sum_{i=1}^{n} \sqrt{\lambda_{1i}}$. Thus (8) is proved.

Let now $B = U_1 A_1 \ldots U_m A_m$ and let $\{\kappa_i\}$ be the eigenvalues of B^*B. Since $(U_j A_j)^*(U_j A_j) = A_j^* A_j$, repeated application of Horn's theorem gives

$$\prod_{i=1}^{n} \kappa_i \leq \prod_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \ldots \lambda_{mi}), \quad (n = 1, 2, \ldots) \tag{9}$$

which, according to a lemma of Polya,[8] implies $\sum_{i=1}^{n} \sqrt{\kappa_i} \leq \sum_{i=1}^{n} (\lambda_{1i}\lambda_{2i} \ldots \lambda_{mi})^{1/2}$. Hence (6) will result when we apply (8) to B.

According to Lemmas 2, 3 in an earlier note[9] by the author, we have for any n orthonormal elements $x_1, x_2, \ldots, x_n$:

$$\left| \operatorname*{Det}_{1 \leq i, k \leq n} (Bx_i, x_k) \right|^2 \leq \operatorname*{Det}_{1 \leq i, k \leq n} (B^* B x_i, x_k) \leq \prod_{i=1}^{n} \kappa_i,$$

which, together with (9), yields (7).

3. THEOREM 2. *Let A, B be two c. c. operators in $\mathfrak{H}$. If $\{\lambda_i\}$, $\{\kappa_i\}$, $\{\mu_i\}$, $\{\sigma_i\}$ are the eigenvalues of A^*A, B^*B, $(AB)^*(AB)$, $(A+B)^*(A+B)$, respectively, then for any two non-negative integers m, n:*

$$\mu_{m+n+1} \leq \lambda_{m+1} \cdot \kappa_{n+1} \tag{10}$$

$$\sqrt{\sigma_{m+n+1}} \leq \sqrt{\lambda_{m+1}} + \sqrt{\kappa_{n+1}}. \tag{11}$$

Proof: Let $\{x_i\}$, $\{y_i\}$ be two complete orthonormal sets such that $AA^*x_i = \lambda_i x_i$, $B^*By_i = \kappa_i y_i$ $(i = 1, 2, \ldots)$. Let $AB = WH$ be the polar factorization of AB, where H is the non-negative square root of $(AB)^*(AB)$ and W is partially isometric. For any z of $\mathfrak{H}$, we have $(Hz, z)^2 = (W^*ABz, z)^2 \leq (AA^*Wz, Wz)(B^*Bz, z)$. If z is subject to conditions $||z|| = 1$, $(z, W^*x_i) = 0$ $(1 \leq i \leq m)$ and $(z, y_j) = 0$ $(1 \leq j \leq n)$, then $(AA^*Wz, Wz) \leq ||Wz||^2\lambda_{m+1} \leq \lambda_{m+1}$, $(B^*Bz, z) \leq \kappa_{n+1}$ and therefore $(Hz, z)^2 \leq \lambda_{m+1} \cdot \kappa_{n+1}$. Thus we get (10) by applying the minimum-maximum principle to H. The proof of (11) is similar.

The next theorem deals with asymptotic distribution of eigenvalues.

THEOREM 3. *Let A, B be two c. c. operators in $\mathfrak{H}$. Let $\{\lambda_i\}$, $\{\kappa_i\}$, $\{\sigma_i\}$ be the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$ respectively. If, for some exponent $r > 0$, we have*

$$\lim_{n\to\infty} n^r\lambda_n = \omega < +\infty \text{ and } \lim_{n\to\infty} n^r\kappa_n = 0, \tag{12}$$

then

$$\lim_{n\to\infty} n^r\sigma_n = \omega. \tag{13}$$

Proof: Let h be a fixed positive integer. By (11), we have $\sqrt{\sigma_{(h+1)n+j}} \leq \sqrt{\lambda_{hn+j}} + \sqrt{\kappa_{n+1}}$ $(n = 1, 2, 3, \ldots;\quad j = 0, 1, \ldots, h)$, which combined with (12), gives

$$\overline{\lim_{n\to\infty}}\, n^{r/2}\,\sigma_n^{1/2} \leq [(h+1)/h]^{r/2} \cdot \omega^{1/2}. \tag{14}$$

If we apply (11) to $A = (A+B) + (-B)$, we get inequality $\sqrt{\sigma_{hn+j}} \geq \sqrt{\lambda_{(h+1)n+j}} - \sqrt{\kappa_{n+1}}$, which together with (12) yields

$$\underline{\lim_{n\to\infty}}\, n^{r/2}\,\sigma_n^{1/2} \geq [h/(h+1)]^{r/2}\,\omega^{1/2}. \tag{15}$$

Then (13) results from (14), (15) by letting $h \to \infty$.

4. We turn now to a general inequality for symmetric gauge functions.[10]

THEOREM 4. *Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$. Then a necessary and sufficient condition that*

$$\Phi(a_1, a_2, \ldots, a_n) \leq \Phi(b_1, b_2, \ldots, b_n) \tag{16}$$

for all symmetric gauge functions Φ of n variables, is that

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \qquad (1 \leq k \leq n). \tag{17}$$

The necessity of (17) is immediate, when we consider, for each k, the particular symmetric gauge function Φ defined by

$$\Phi(\xi_1, \xi_2, \ldots, \xi_n) = \operatorname*{Max}_{i_1<i_2<\ldots<i_k} (|\xi_{i_1}| + |\xi_{i_2}| + \ldots + |\xi_{i_k}|).$$

For the proof of the sufficiency of (17), we only indicate that it results from the following two lemmas.

LEMMA 2. *Let $\{b_i\}$ be a set of n real numbers. Let $0 < \alpha < 1$ and let h, j be two distinct fixed indices. If*

$$c_h = \alpha b_h + (1 - \alpha)b_j, \qquad c_j = (1 - \alpha)b_h + \alpha b_j, \qquad c_i = b_i \text{ for } i \neq h, j, \tag{18}$$

then $\Phi(c_1, c_2, \ldots, c_n) \leq \Phi(b_1, b_2, \ldots, b_n)$ for all symmetric gauge functions Φ of n variables.

LEMMA 3. *Let $\{a_i\}$, $\{b_i\}$ be two sets each of n real numbers satisfying (17) and $a_i \geq a_{i+1}$. Then by the successive application of a finite number of substitutions of form (18), $\{b_i\}$ can be reduced to a set $\{d_i\}$ such that $a_i \leq d_i$ $(1 \leq i \leq n)$.*[11]

5. THEOREM 5. *Let A, B be two c. c. operators in $\mathfrak{H}$. If $\{\lambda_i\}$, $\{\kappa_i\}$ and $\{\sigma_i\}$ are the eigenvalues of A^*A, B^*B and $(A + B)^*(A + B)$ respectively, then*

$$\Phi(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_n}) \leq \Phi(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}) + \Phi(\sqrt{\kappa_1}, \sqrt{\kappa_2}, \ldots, \sqrt{\kappa_n}) \tag{19}$$

holds for every symmetric gauge function Φ of any number n of variables.

Proof: By Theorem 4, we need only to prove the following special case of (19):

$$\sum_{i=1}^{n} \sqrt{\sigma_i} \leq \sum_{i=1}^{n} \sqrt{\lambda_i} + \sum_{i=1}^{n} \sqrt{\kappa_i}. \qquad (n = 1, 2, 3, \ldots). \tag{20}$$

Consider the polar factorization of $A + B = WH$, where H is the non-negative square root of $(A + B)^*(A + B)$ and W is partially isometric. Let $z_1, z_2, \ldots, z_n$ be n orthonormal elements such that $Hz_i = \sqrt{\sigma_i} z_i$ $(1 \leq i \leq n)$. We can find a unitary operator U such that $U(A + B)z_i = W^*(A + B)z_i = Hz_i$ $(1 \leq i \leq n)$. Then

$$\sum_{i=1}^{n} \sqrt{\sigma_i} = \sum_{i=1}^{n} (U(A + B)z_i, z_i) = \sum_{i=1}^{n} (UAz_i, z_i) + \sum_{i=1}^{n} (UBz_i, z_i)$$

and (20) follows from

$$\left|\sum_{i=1}^{n}(UAz_i, z_i)\right| \leq \sum_{i=1}^{n}\sqrt{\lambda_i}, \qquad \left|\sum_{i=1}^{n}(UBz_i, z_i)\right| \leq \sum_{i=1}^{n}\sqrt{\kappa_i},$$

which are true by (4).

* This work was supported in part by the Office of Naval Research.

[1] Von Neumann, J., "Some Matrix-Inequalities and Metrization of Matric-Space," *Tomsk Univ. Rev.*, **1**, 286–300 (1937).

[2] The eigenvalues will always be arranged in order of decreasing absolute value.

[3] Weyl, H., "Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," these PROCEEDINGS, **35**, 408–411 (1949).

[4] Horn, A., "On the Singular Values of a Product of Completely Continuous Operators," *Ibid.*, **36**, 374–375 (1950).

[5] Weyl, H., "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen," *Math. Annalen*, **71**, 441–479 (1912).

[6] Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge University Press, Cambridge, 1934, pp. 44–49.

[7] Von Neumann, J., "Über adjungierte Funktionaloperatoren," *Ann. of Math.*, **33**, 294–310 (1932).

[8] Polya, G., "Remark on Weyl's Note: Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," these PROCEEDINGS, **36**, 49–51 (1950).

[9] Fan, K., "On a Theorem of Weyl concerning Eigenvalues of Linear Transformations, II," *Ibid.*, **36**, 31–35 (1950).

[10] Following von Neumann, a gauge function Φ (in the sense of Minkowski) is called *symmetric*, if $\Phi(\xi_1, \xi_2, \ldots, \xi_n) = \Phi(\epsilon_1\xi_{j_1}, \epsilon_2\xi_{j_2}, \ldots, \epsilon_n\xi_{j_n})$ for any combination of signs $\epsilon_i = \pm 1$ and for any permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$. For general properties of symmetric gauge functions, see Schatten, R., *A Theory of Cross-Spaces*, Princeton University Press, Princeton, 1950, pp. 84–92.

[11] The proof of Lemma 3 is similar to the corresponding part of the proof of Muirhead's theorem (see the reference in footnote 6). Using Lemma 3, one can also prove: For two sets $\{a_i\}$, $\{b_i\}$ each of n non-negative numbers arranged in descending order, condition (17) is equivalent to the existence of n^2 non-negative numbers p_{ij} satisfying (1) and $a_i = \sum_{j=1}^{n} p_{ij}b_j$ $(1 \leq i \leq n)$.

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*INVARIANT DIFFERENTIAL FORMS IN SEVERAL GROUP VARIABLES AS SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS IN FRÉCHET DIFFERENTIALS**

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Communicated by G. C. Evans, September 4, 1951

1. Introduction.—There is a particularly interesting class of partial differential equations in Fréchet differentials that characterize abstract invariants and in particular differential forms that are left invariant by the parameter groups π_1 and π_2 of a differentiable abstract group Δ. We shall

concern ourselves with alternating differential forms in several group variables invariant under π_1 and/or π_2 of an abstract group Δ imbedded in a Banach space B. These groups Δ are topological groups but they are not necessarily compact nor even locally compact. Hence the Haar measure theory is not available here in general. Of course if Δ is assumed to be compact or locally compact, then Haar measure theory is applicable and leads to useful results in invariant theory.

The abstract group Δ will be assumed to be an open set imbedded in a Banach space such that the *group product* $\alpha\beta$ and *inverse* α' together with the *unit* ϵ form a group of Class C_2 (i.e., the group functions $\alpha\beta$ and α' have continuous second successive Fréchet differentials throughout Δ). Other assumptions on Δ will be explicitly stated. A Fréchet differential of a function $f(\alpha)$ with arguments and values in Banach spaces will be denoted usually by $f(\alpha;\ y)$ if the increment is y and the differential is at $\alpha = \alpha$. A partial Fréchet differential in the ith variable of a function $f(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ will be denoted by $f_{\alpha_i}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n;\ y)$ if y is the increment and the evaluation is at $\alpha_i = \alpha_i$. In the special case of a partial Fréchet differential in the *first* variable, we shall often write more simply $f(\alpha, \beta_1, \ldots, \beta_n;\ y)$ for the Fréchet differential in α at $\alpha = \alpha$ with increment y of the function $f(\alpha, \beta_1, \ldots, \beta_n)$.

The parameter groups π_1 and π_2 are respectively the continuous transformation groups

$$\begin{aligned} \pi_1&: \quad \bar{\alpha} = \alpha\beta \qquad (\beta \text{ is the parameter}),\\ \pi_2&: \quad \bar{\beta} = \alpha\beta \qquad (\alpha \text{ is the parameter}. \end{aligned}$$

Let us define two functions

$$\begin{aligned} f(\sigma, \alpha, \gamma) &= {}^{\text{def.}}\sigma'\alpha\gamma\\ g(\gamma, \beta, \sigma) &= {}^{\text{def.}}\gamma\beta\sigma' \end{aligned}$$

where γ is an arbitrary but fixed element of Δ while α, β and σ are independent variables over Δ. The two linear differential forms

$$\left.\begin{aligned} \Omega(\beta, \delta\beta) &= [f(\sigma, \beta, \gamma; \delta\beta)]_{\sigma=\beta},\\ \Phi(\alpha, \delta\alpha) &= [g_\sigma(\gamma, \alpha, \sigma; \delta\alpha)]_{\sigma=\alpha} \end{aligned}\right\} \tag{1.1}$$

can be shown to be solvable linear functions of their second variables with inverses $\Omega^{-1}(\beta, \xi)$ and $\Phi^{-1}(\alpha, \eta)$ respectively. It is to be noticed that the notation -1 does not mean group inverse. In fact, the values of the functions Ω^{-1} and Φ^{-1} are not in general in the group Δ. It can be shown that the differential forms $\Omega(\beta, \delta\beta)$ and $\Phi(\alpha, \delta\alpha)$ are invariant under π_2 and π_1 respectively for all β, α of the group Δ. Furthermore, they satisfy the following differential equations in Fréchet differentials[1] (the generalizations of the Maurer-Cartan[2] equations of the Lie theory)

$$\Omega(\beta, \delta_1\beta; \delta_2\beta) - \Omega(\beta, \delta_2\beta; \delta_1\beta) = C[\Omega(\beta, \delta_1\beta), \Omega(\beta, \delta_2\beta)] \quad (1.2)$$

and a similar equation for $\Phi(\beta, \delta\beta)$. The abstract alternating bilinear function $C(\xi, \eta)$ is the generalization of the structural constants[2] of the Lie theory.

Finally, to shorten the statements of our theorems, we need to define one more concept and make clear a notation for differential operators. A group Δ will be called *d-arcwise connected* if Δ is arcwise connected with arc functions that have continuous derivatives.

If $A(\beta, \xi)$ is a fixed function of Class C_1 on ΔB to B, then we *define the differential expression* $X_\xi F$ as

$$X_\xi F = \left.\begin{aligned} &\sum_{i=1}^{2} F_{\beta_i}(\beta_1, \beta_2, y, z_1, z_2; A(\beta_i, \xi)) + F(\beta_1, \beta_2, A(\beta_1, \xi; y), z_1, z_2) \\ &+ F(\beta_1, \beta_2, y, A(\beta_2, \xi; z_1), z_2) + F(\beta_1, \beta_2, y, z_1, A(\beta_2, \xi; z_2)), \end{aligned}\right\} \quad (1.3)$$

where $F(\beta_1, \beta_2, y, z_1, z_2)$ is on $\Delta^2 B^3$ to B_1, trilinear in y, z_1 and z_2, and B_1 is a Banach space, not necessarily distinct from B, while $\Delta^2 B^3$ denotes that β_1, β_2 range over Δ and y, z_1, z_2 over B.

2. A Typical Theorem on Invariant Two-Point Differential Forms.—The following is a representative theorem. We believe that both the results and methods of proof are novel even if the Banach spaces B and B_1 reduce to finite dimensional Euclidean number spaces (the same can be said for Theorem II of §3). Only a few remarks on the method of proof can be given here as the complete proof is long. The results are global, i.e., in the large.

THEOREM I. *Let $\Delta \subset B$, a Banach space, be an open d-arcwise connected group of class C_2. A necessary and sufficient condition that a two-point differential form $W(\beta_1, \beta_2, \delta\beta_1, \delta_1\beta_2, \delta_2\beta_2)$ of class C_1 (for β_1, β_2 in Δ, and $\delta\beta_1$, $\delta_i\beta_2$ in B) with values in a Banach space B_1, not necessarily distinct from B, linear in $\delta\beta_1$ and alternating bilinear in $\delta_1\beta_2$, $\delta_2\beta_2$, be invariant under the second parameter group π_2 of Δ is that W satisfy the following partial differential equation in Fréchet differentials*

$$X_\xi W = 0 \quad (2.1)$$

with the understanding that the $\Phi^{-1}(\beta, \xi)$ defined in connection with (1.1) is the fixed function $A(\beta, \xi)$ in $X_\xi W$.

For the sufficiency part of the proof, it is not difficult to establish first that there exists a function $\lambda(\beta_1, \beta_2, \xi_1, \eta_1, \eta_2)$ on $\Delta^2 B^3$ to B_1, linear in ξ_1, and alternating bilinear in η_1 and η_2 such that

$$W(\beta_1, \beta_2, \delta\beta_1, \delta_1\beta_2, \delta_2\beta_2) = \lambda(\beta_1, \beta_2, \Omega(\beta_1, \delta\beta_1), \Omega(\beta_2, \delta_1\beta_2), \Omega(\beta_2, \delta_2\beta_2)). \quad (2.2)$$

The theory of the abstract Pfaffian differential equation[3]

$$P(\beta, \xi; \delta\beta) = C(P(\beta, \xi), \Omega(\beta, \delta\beta)) \quad (2.3)$$

satisfied by $P(\beta, \xi) = \Omega(\beta, \Phi^{-1}(\beta, \xi))$ for all $\beta \epsilon \Delta$ and $\xi, \delta\beta \epsilon B$, and of the abstract[3] equation (1.2) are used to show that $\lambda(\beta_1, \beta_2, \xi_1 \eta_1, \eta_2)$ must be invariant under π_2 as a function of the group variables β_1 and β_2 whenever (2.2) satisfies the equation (2.1). Finally it is shown that

$$W = \lambda(\beta_1'\beta_2, \Omega(\beta_1, \delta\beta_1), \Omega(\beta_2, \delta_1\beta_2), \Omega(\beta_2, \delta_2\beta_2)) \tag{2.4}$$

for some function $\lambda(\beta, \xi, \eta_1, \eta_2)$ of only *four* variables. From this the sufficiency part of the condition follows without much difficulty. The proof for the necessity part of the condition uses a method that essentially reverses the steps for the sufficiency proof but does not use the d-arcwise connectedness of the group Δ.

There is a corresponding theorem for invariance under π_1 with the corresponding partial differential equation

$$X_\xi W = 0 \tag{2.5}$$

in which $A(\beta, \xi)$ is now replaced by $\Omega^{-1}(\beta, \xi)$.

3. A Typical Theorem on Invariance Under Both Parameter Groups.—Necessary and sufficient conditions that a differential form W be invariant under both π_1 and π_2 can clearly be given in terms of W *satisfying the pair of partial differential equations* (2.1) *and* (2.5). However, there is another type of theorem proved by the writer in which the necessary and sufficient condition for invariance is not in terms of solutions of partial differential equations. For example, *a necessary and sufficient condition* (*under the restrictions of Theorem I*) *that* W *be invariant under both* π_1 *and* π_2 *is that* W *be of form* (2.4), *that* $\lambda(\beta, \xi, \eta_1, \eta_2)$ *be a class function of the group variable* β *satisfy the functional equation*

$$\lambda(\beta, C(\xi_1, \xi), \eta_1, \eta_2) + \lambda(\beta, \xi_1, C(\eta_1, \xi), \eta_2) + \lambda(\beta, \xi_1, \eta_1, C(\eta_2, \xi)) = 0 \tag{3.1}$$

for all $\beta \epsilon \Delta$ *and* $\xi, \xi_1, \eta_1, \eta_2 \epsilon B$.

There is still another type of theorem that combines some of the features of the last theorem together with those of Theorem I. We shall state it as

THEOREM II. *Under the restrictions of Theorem I, a necessary and sufficient condition that the two-point differential form* W *be invariant under both parameter groups* π_1 *and* π_2 *of the group* Δ *is that* W *satisfy the abstract partial differential equation* (2.1) *and that the function* λ *defined by*

$$\lambda(\beta, \xi_1, \eta_1, \eta_2) = W[\epsilon, \beta, \Omega^{-1}(\epsilon, \xi_1), \Omega^{-1}(\beta, \eta_1), \Omega^{-1}(\beta, \eta_2)]$$

(ϵ, *the unit of* Δ) *be a class function of* β *and satisfy the functional equation* (3.1).

Now, in the theory of abstract *one-point* alternating differential forms invariant under *both* π_1 and π_2 (in the interim of publication[3]), *the abstract*[4]

Poincaré-Stokes differential of the form is the zero element of the value Banach space B_1. It can be shown by counter examples that such is *not* in general the case for *multiple-point* forms invariant under both π_1 and π_2, i.e., the abstract Poincaré-Stokes differential may not be zero with respect to any of the group points.

4. Concluding Remarks.—We mention that the theory of analytical functions of abstract[5,6] variables in complex Banach spaces (as well as real Banach spaces) has immediate application in case we take the spaces B and B_1 to the complex Banach spaces. One can also consider invariant multiple-point Hermitean differential forms.

The author's differential calculus[7] for functions of topological abelian group variables can be used in generalizing the subject of abstract differentiable groups and their invariant differental forms whenever general topological abelian groups are taken in the place of the Banach spaces. Some theorems that involve d-arcwise connectedness would need modification. If the group Δ is imbedded in a locally convex linear topological space, then the theorems of this note continue to hold with the understanding that the notion of a function of Class C_n is made a little more restrictive.

In conclusion, we wish to mention that these studies on invariants have connections with harmonic forms and their generalizations, classical as well as modern differential geometry, and the author's general differential geometries with abstract coordinates.[8]

* This note represents a summary of a chapter of an extensive monograph in preparation: Michal, A. D., *Differential Equations in Abstract Spaces with Applications*. We shall refer to this by the abbreviation *DEASA*. This note can also be considered as a continuation of the author's early studies on non-additive integral invariants: Michal, A. D., Functionals of R-Dimensional Manifolds Admitting Continuous Groups of Point Transformations, *Trans. Am. Math. Soc.*, **29**, 612–646 (1927).

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[2] Cartan, Elie, *La Théorie Des Groupes Finis Et Continus Et La Géométrie Differentielle*, Gauthier-Villars, Paris, 1937; Eisenhart, L. P., *Continuous Groups of Transformations*, Princeton Univ. Press, 1933.

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*REPRESENTATION OF NUCLEON-NUCLEON SINGLET S SCATTERING**

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Communicated September 27, 1951

In early work[1] on proton-proton scattering it has been found convenient to deal with the function

$$f = \{2\pi/[-1 + \exp(2\pi\eta)]\} \cot K_0 + 4\gamma - 2 + 2\text{R.P.}[\Gamma'(i\eta)/\Gamma(i\eta)] - 2\ln\eta$$

where

$$\eta = ZZ'e^2/\hbar v,$$
$$\gamma = 0.57721\ldots = \text{Euler-Mascheroni constant},$$

K_0 is the phase shift, Z,Z' are atomic numbers of the colliding particles and $\Gamma(x)$ is the gamma function of x. The special reason for introducing the function f was the fact that the logarithmic derivative of $rR = F$, where R = radial function, at the outer boundary of a short range potential well is expressible in terms of f and quantities which are power series in the energy E. For potentials of very short range the function f is[1] a constant and the slope of f plotted against E is proportional to the range of force as is clear[1] from the formulas connecting f with the logarithmic derivative and other quantities. While the function f and its approximate linearity have been used as intermediate steps in the earlier calculations a strong impetus to its adoption in recent years has been given by Schwinger[2] who found it convenient to deal with this quantity by a process of conversion of the radial differential equation to an integral equation and representing f as a variational integral. Following Schwinger, Blatt[2] and Jackson and Blatt[2] have employed f and its limiting form for $ZZ'e^2 = 0$ as a convenient means of analyzing experimental material while Breit and Bouricius[3] have made similar studies without the variational principle. Other treatments of the simpler properties of f have appeared[4] since; in these the variational method is avoided and the first two terms of f are dealt with directly by means of the differential equation. The impression has nevertheless been created that the variational method puts f in its true ("natural") light and that its power series representation can be studied by means of the variational

method more simply than otherwise. It is apparent, on the other hand, that in a problem which amounts to a study of an ordinary differential equation of second order all methods must have rather immediate connections. It is the modest purpose of the present note to describe an approach in which the interrelations of the variational and direct treatments can be seen in a simple manner and to point out a combined use of the two ways if one wishes to obtain a simple expression for the general term in the expansion of f in powers of E or else in powers of a linear parameter such as a change in depth of a potential energy well.

It follows from the stationary property of the variational integral that an error in the wave function of relative order ϵ produces an error in f of relative order ϵ^2 or higher. This fact is readily seen to indicate the existence of identities between integrals containing coefficients of E^j in the expansion of F in powers of E. The argument from here on is essentially one regarding the form of the answer rather than the degree of approximation. It is convenient to start out with the slight generalization of Schwinger's form for $\partial f/\partial E$ which has been used by Breit and Hatcher,[4] viz.

$$\partial(f - f')/\partial\kappa = a \int_0^b (u'^2 - u^2)\, dr \qquad (1)$$

where b is the radius of the potential well, f' is f for a comparison potential, κ is k^2 where k is the usual 2π divided by wave-length, $a = \hbar^2/\mu e^2 ZZ'$ is the Bohr length with the reduced mass denoted by μ and u, u' are suitably renormalized functions F for the actual and the comparison potentials. The normalization of u, u' is such that if the function is continued as the solution of the Coulomb wave equation to $r = 0$ the value of the function is unity. The form of equation (1) is identical with that of Schwinger but it is used here for a variable $E \neq 0$. One expands

$$u = \sum_0^\infty (\kappa - \kappa_0)^n u_n \qquad (1.1)$$

and obtains from (1)

$$a \sum_n (\kappa - \kappa_0)^n \sum_{i+j=n} \int_0^b (u'_i u'_j - u_i u_j)\, dr = \sum_n n(\kappa - \kappa_0)^{n-1}[f^{(n)} - f'^{(n)}], \qquad (1.2)$$

where

$$f = \sum_n (\kappa - \kappa_0)^n f^{(n)},$$

and a similar formula for f' define the expansion coefficients $f^{(n)}$, $f'^{(n)}$. It follows from equation (1.2) that

$$n[f^{(n)} - f'^{(n)}] = a \sum_{i+j=n-1} \int_0^b (u'_i u'_j - u_i u_j)\, dr. \qquad (2)$$

On the right side of this formula there enter all the expansion coefficients $u_0, u_1, \ldots, u_{n-1}$. The variational principle shows, however, that if $n = 2\nu + 1$ is odd then the knowledge of $u_0, \ldots, u_\nu$ suffices for the determination of $f^{(0)}, \ldots, f^{(2\nu+1)} = f^{(n)}$ so that the entrance of $u_{\nu+1}, \ldots u_{n-1}$ in equation (2) is unnecessary. Similarly if $n = 2\nu$ is even then the knowledge of $u_0, \ldots u_\nu$ should suffice and the entrance of $u_{\nu+1}, \ldots u_{n-1}$ is unnecessary. There must exist, therefore, identities between the integrals in equation (2). From here on one does not need the variational principle because one finds from the differential equation that

$$(s - 1, t + 1) = (s, t), \tag{2.1}$$

where

$$(s, t) = \int_0^b (u'_s u'_t - u_s u_t)\, dr. \tag{2.2}$$

It is thus seen that

$$\begin{aligned} f^{(n)} - f'^{(n)} &= a(s, s), \ (n = 2s + 1), \\ f^{(n)} - f'^{(n)} &= a(s - 1, s), \ (n = 2s), \end{aligned} \tag{2.3}$$

which is a general answer to the problem of obtaining the coefficients in the expansion of f in powers of E. For a linear parameter λ entering the differential equation as in

$$d^2u/dr^2 + [\kappa + \lambda\varphi(r) + \ldots]u = 0 \tag{3}$$

one obtains

$$\partial f/\partial\lambda = -a \int_0^b \varphi u^2\, dr. \tag{3.1}$$

The introduction of the symbol

$$[s, t]_\lambda = \int_0^b u_{s\lambda} u_{t\lambda} \varphi\, dr \tag{3.2}$$

yields, by an argument very similar to that for $\kappa - \kappa_0$,

$$f = f_0 + \lambda f_{1\lambda} + \lambda^2 f_{2\lambda} + \ldots, \tag{4}$$

$$\begin{aligned} f_{n\lambda} &= -a[s, s], \qquad (n = 2s + 1); \\ f_n &= -a[s - 1, s], \ (n = 2s). \end{aligned} \tag{4.1}$$

The above discussion applies to other problems than the nucleon-nucleon singlet S scattering although the more important applications of the f function have been so far principally in this field. It is seen that the technique of calculation is simpler by the direct approach but that the stationary property of the variational integral, while not indispensable, is nevertheless a valuable guide.

* This paper has been read at the Washington Meeting of the American Physical Society, Spring, 1951.

† Assisted by the joint program of the ONR and AEC.

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